Math 231br (Advanced Algebraic Topology) Lecture Notes

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Contents

1 Introduction

Did not attend as it was administrivia. Allow me to use this space to give some general information.

These lecture notes are taken during Spring 2015 for Math 231br (Advanced Algebraic Topology) at Harvard. The course was taught by Professor Michael Hopkins. The course is a continuation of Math 231a, which covers the first three chapters of Allan Hatcher's *Algebraic Topology* (henceforth referred to as simply "Hatcher"). The course is self-contained in that it does not assume background beyond that of Math 231a. However, due to time constraint, the last part regarding cobordism consisted largely of proof sketches, and a lot of details were skipped. Thus for a deeper understanding of the material, reading one of the "core references" given below alongside this note is highly recommended.

Regarding spectral sequences: in the middle of notetaking I discovered the preferable `sseq` package, so switched to it. As a result you'll see two ways of drawing spectral sequences in this document. I may unify them in the future, but since drawing spectral sequence is a time-consuming task, I can't make a promise.

Below I list the literature used/mentioned throughout this course:

- Background: Review of basic properties of fundamental groups, singular homologies and cohomologies.

    Algebraic Topology by Allan Hatcher

- Core References: Contain proofs presented in this note, along with more details.

    Vector Bundles and K Theory by Allan Hatcher

    Cohomology Operations and Applications in Homotopy Theory by Robert E. Mosher

    A Concise Course in Algebraic Topology by J. P. May

- Other References: Other materials mentioned throughout the course. Collected here for convenience.

    Homotopy Theories and Model Categories by W.G. Dwyer and J. Spalinski

    Cohomology Operations by Steenrod and Epstein

    Notes on Cobordism Theory by Robert E. Stong

I've added some comments throughout this note, and they are all in the following format:
(This is a comment — Charles).

These notes may still contain numerous errors and typos, so please use at your own risk. I take responsibility for all errors in the notes. There are some references here and there to the homework problems, which can also be found on my website.

2 Higher Homotopy Groups

Let $(X, *)$ be a pointed space. $\pi_n(X) = \{f : S^n \to X \mid f(*) = *\}$ modulo pointed homotopy. Sometimes written as $[S^n, X]_*$. $\pi_0(X)$ is the set of path components. $\pi_1(X)$ is the fundamental group. So how do we describe $\pi_2(X)$? In particular how does it have a group structure? One straightforward way is $S^2 \to S^2 \wedge S^2 \to X$ (glue f and g by $f \wedge g$), but there are other ways,

2.1 Hurewicz's Construction of Higher Homotopy Groups

For instance, the following given by Hurewicz. Start with the set-theoretical part that $\pi_N(X) = \{f : I^n \to X \mid f(\partial I^n) = *\}$ modulo homotopy. Then glue the I^ns together as "adjacent cubes" gives the group structure.

Let A, X be topological spaces. Denote by X^A the set of maps from A to X. One would like to topologize X^A such that continuous map $Z \to X^A$ is exactly the same as continuous map $Z \times A \to X$. This can be done by giving it the compact-open topology. Some point-set topology issues arise due to the fact that the spaces A and X are not necessarily compact Hausdorff, and is resolved by using compactly generated spaces (c.f. *Steenrod, Convenient category of topological spaces*).

Suppose A, Z, X have base points $*$. We only want to look at pointed maps i.e. to look at $(X, *)^{(A,*)}$. We make this into a pointed space by assigning the base point $* := A \to *$. Now consider $(Z, *) \to (X, *)^{(A,*)}$. This then corresponds to a special map $Z \times A \to X$ that factors through $Z \times A/(* \times A \cup Z \times *)$, i.e. the smash product $Z \wedge A$. Make $Z \wedge A$ into a pointed space by assigning $* := \{Z \times * \cup * \times A\}$. Then $(Z, *) \to (X, *)^{(A,*)}$ corresponds to $(Z \wedge A, *) \to (X, *)$.

Concretize this with $(I^n, \partial I^n) = S^n = (I, \partial I) \wedge (I^{n-1}, \partial I^{n-1}) \to (X, *)$, then we see it corresponds to $I/\partial I \to (X, *)^{(I^{n-1}, \partial I^{n-1})}$. The latter is called $\Omega^n(X)$, the n-fold loop space. More concretely, $\Omega(X) = Map((I, \partial I), (X, *)) = Map((S^1, *), (X, *))$, and $\Omega^n(X) = \Omega^{(n)}(X)$. Now Hurewicz's definition for higher homotopy groups is simply $\pi_n(X) = \pi_1(\Omega^{n-1}(X))$. One can easily generalize this into $\pi_n(X) = \pi_p(\Omega^q(X))$, where $p + q = n$.

Uniqueness and Commutativity There are certainly two ways of "gluing": f to the left of g (which we denote by $f * g$) and f on top of g (which we denote by $f \circ g$). Then if we apply this to a 2-by-2 square, we see that one have $(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$. Now the Eckmann-Hilton argument tells us that the two operations are equal and both commutative. In particular we know that $\pi_n(X)$ is abelian for $n > 1$.

2.2 Relative Higher Homotopy Groups

For $n > 0$ and A a subspace of X, define $\pi_n(X, A) = Map((D^n, \partial D^n, *), (X, A, *))$ modulo homotopy. Then we get a long exact sequence $\pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \ldots \to \pi_0(A) \to \pi_0(X)$ is exact. (Some cautions need to be taken near the end of the exact sequence as to how to interpret it; in particular, starting from $\pi_1(X, A)$ we no longer have groups, but only pointed sets.)

Problem 1. *Let $\partial I^n = I^{n-1} \cup J^{n-1}$. Clearly J^{n-1} is contractible and we have $J^{n-1} \subseteq \partial I^n \subseteq I^n$. We can also define $\pi_n(X, A) = Map((I^n, \partial I^n, J^{n-1}), (X, A, *))$ modulo homotopy. Can we write $\pi_n(X, A)$ as a homotopy group of some other space? (Hint: it is π_{n-1} of something else.)*

Answer This is not hard—it is π_{n-1} of the path space $P(X, A)$, which is a pointed space based on the family of paths $I \to X$ that begins at $*_X$ and ends in A, equipped with the compact-open topology and has the base point $*_{P(X,A)} := I \to *_X$. Let's be more rigorous about this. Note that $P(X, A)$ is the same as $Map((I^1, \partial I^1, J^0), (X, A, *))$. Let the base point be denoted as above. Then by directly expanding $(I^{n-1}, \partial I^{n-1}) \to P(X, A)$ one sees that this corresponds to $I^n \to X$ such that $\partial I^{n-1} \times I^1 \cup I^{n-1} \times J^0 = J^{n-1}$ goes to $*_X$, and $I^{n-1} \times I^0 \to A$. Thus it corresponds to the definition above; the other way around is just formalism and we skip it.

3 Hurewicz Fibration, Serre Fibration and Fiber Bundle

Did not attend due to personal schedule conflict. I wrote the following notes to cover the material gap. Throughout this discussion, let E, B be basepointed topological spaces. For the sake of simplicity I'm not including all proofs; most of the proofs can be found in Hatcher, or this online note.

3.1 Definitions

Definition 1 (Homotopy Lifting Property). *A map $p : E \to B$ is said to have the homotopy lifting property with respect to X if for any f, g in the following diagram, the diagonal map $h : X \times I \to E$ in the diagram below always exists:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

where I is the unit $[0, 1]$ and i is the inclusion $x \mapsto (x, 0)$.

Definition 2 (Relative Homotopy Lifting). *$p : E \to B$ is said to have homotopy lifting property with respect to a pair (X, A) (where $A \subseteq X$) if the diagonal map h exists for any f, g and any given homotopy $H : A \times I \to E$:*

$$\begin{tikzcd} X \cup A \times I \arrow[r, "f \cup H"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

where i is the obvious inclusion and we're promised that $H(a, 0) = f(a)$ for any $a \in A \subseteq X$.

The intuition is straightforward: to say $p : E \to B$ has HLP with respect to (X, A) is to say that any homotopy of $X \to B$ can be lifted to a homotopy $X \to E$, provided that a homotopy already exists on A.

Definition 3 (Hurewicz and Serre Fibrations). *We say $p : E \to B$ is a* Hurewicz *fibration if it has HLP for all space X, and say it is a* Serre *fibration if it has HLP for all CW complex X.*

While we're on this, let's also define the dual notion for HLP and Hurewicz fibration, although we won't mention them again until the end of this lecture. (The correct dual notion for Serre fibration, sometimes called "the Serre cofibration," is the retract of relative cell complexes; see Lecture 13 for further discussions.)

Definition 4 (Homotopy Extension Property). *Let $i : A \to X$ be a mapping of topological spaces. It satisfies the* homotopy extension property *with respect to some space Y if, given f and h_0 in the following diagram:*

$$\begin{tikzcd} Y & X \arrow[l, "h_0"] \arrow[dl, dashed, "h"] \\ Y^I \arrow[u, "g \mapsto g(0)"] & A \arrow[l, "f"] \arrow[u, "i"'] \end{tikzcd}$$

the diagonal map h always exist.

Definition 5 (Cofibration). *A mapping $i : A \to X$ is called a* cofibration *if it satisfies HEP with respect to any space Y.*

The concept of a fibration is closely related to that of a section, as the next fact shows.

Lemma 1. *Suppose $p : E \to B$ is a Hurewicz (resp. Serre) fibration, and that we have a subspace pair (resp. CW pair) (X, A) where the inclusion is $i : A \to X$. Given the following diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

If either i or p is a homotopy equivalence (resp. weak equivalence), then h exists.

Proof. The Hurewicz case is trivial; for the Serre case one needs to inductively construct h one dimension at a time, then use the standard pasting trick (doubling the speed of homotopy at each higher dimension) to obtain h. See Theorem 5.1 of this note. ∎

Proposition 1 (Existence of Sections)**.** *If p is a nonempty Hurewicz (resp. Serre) fibration, and B is a contractible space (resp. CW complex), then p admits a section, i.e. there is another map g such that $p(g(x)) = x$ for all $x \in B$.*

Proof. Take $X = B$, $g = 1_B$, $A = * \in B$, f anything in the lemma above. $\square$

There are some other ways to define a Serre fibration which can be useful at times. Let us record them down here.

Proposition 2. *The following are equivalent:*

1. *$p : E \to B$ is a Serre fibration.*
2. *$p : E \to B$ has HLP for all disks.*
3. *$p : E \to B$ has HLP for all pairs (D^n, S^{n-1}).*
4. *$p : E \to B$ has HLP for all CW pairs (X, A).*

Proof. For 2 to 3, note that $(D^n \times I, D^n \times * \cup S^{n-1} \times I)$ is homeomorphic to $(D^n \times I, D^n \times *)$. For 3 to 4, do induction on the cell structure. The rest is trivial. $\square$

Example 1. *The trivial fibration, $E = B \times F$ for some space F, and p being the projection onto the first factor, is obviously a Hurewicz fibration.*

Definition 6 (Fiber Bundles)**.** *Given a Serre fibration $p : E \to B$, if in addition there is a space F called a* fiber*, such that we have the following local triviality condition: for each $e \in E$, there is a neighborhood U_e of $p(e)$ in B, such that there is a homeomorphism $\varphi : p^{-1}(U_e) \to U_e \times F$ that makes the following diagram commute:*

$$\begin{tikzcd}
p^{-1}(U_e) \arrow[r, "\varphi"] \arrow[d, "p"] & U_e \times F \arrow[dl, "\pi_1"] \\
U_e &
\end{tikzcd}$$

where π_1 is the projection onto the first factor, then we say that p is a fiber bundle*.*

Obviously, $p^{-1}(x)$ is homeomorphic to F for any $x \in B$, so we say F is *the* fiber of p, and sometimes write p as $F \to E \to B$. One can easily check the following:

Lemma 2. *Any compositions and pullbacks of Hurewicz (resp. Serre) fibrations are again Hurewicz (resp. Serre) fibrations.*

Remark 1. *Though the pullback part is still true, it might be worth pointing out that the composition part is no longer true for fiber bundles; however, under certain conditions (such as in the category of finite-dimensional manifolds, with smooth fiber bundles), we may still have the composition property. See here for a general discussion.*

3.2 Relationship Among Them

Some more words regarding the relationship among the three concepts.

Lemma 3. *Being a Serre fibration is a local property; more concretely, suppose $p : E \to B$ is a map such that every point b has a neighborhood U such that $p|_{p^{-1}(U)}$ is a Serre fibration. Then p is a Serre fibration.*

Proof. Theorem 11 of this note. $\square$

Corollary 1. *Every fiber bundle is a Serre fibration.*

Proof. A trivial bundle is a Serre fibration; fiber bundles are locally trivial. $\square$

Example 2. *It turns out if B is paracompact, a fiber bundle would also be a Hurewicz fibration. But here's a fiber bundle that is not a Hurewicz fibration. (source) Let $R_2 = \mathbb{R} \times \{0, 1\}$, and let $X = R_2/\{(x, 0) \sim (x, 1) \forall x \in \mathbb{R}^+\}$, so it's two copies of the real line glued together on the positive half. Let the (images of the) two copies of $\mathbb{R}$ in X be called U and V. Define $f((x, i)) = x$ be the "coordinate projection", and note that it descends down to a map $f : U \cap V \to \mathbb{R}^+$, which cannot be extended to any continuous $X \to \mathbb{R}^+$. (Note that g trivially descends to $X \to \mathbb{R}$.) Define E to be the fiber bundle that is a trivial $\mathbb{R}^+$ bundle on both U and V, and the glueing function is one such that $(x, a) = (x, f(x)a)$ where the LHS is over U and RHS over V. Now suppose* *a section $X \to E$ exists; then in particular it becomes a section on subbundles, and by projecting to the second factor we obtain two functions $g_U : U \to \mathbb{R}^+$ and $g_V : V \to \mathbb{R}^+$ such that $g = g_V/g_U$ is a function that agrees with f on $U \cap V$. However, it is possible to extend g to entire X: just let $g(x) = g_V(j_V(x))/g_U(j_U(x))$ where j_V is first map to the coordinate by g then send to the corresponding point on V, and j_U is defined similarly. This yields a contradiction, so p cannot admit a section and is therefore not a Hurewicz fibration.*

Example 3. *And now an example of a Serre fibration that is not a fiber bundle. Let $E = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1\}$ and $B = I$; let p be the projection onto the first factor.*

So the inclusion order is Hurewicz $\Rightarrow$ Fiber Bundle $\Rightarrow$ Serre.

3.3 Some Properties

Theorem 3.1 (Long Exact Sequence of Serre Fibrations)**.** *Let $p : E \to B$ be a Serre fibration, and let F be the basepoint fiber, then there is a (natural) long exact sequence:*

$$\ldots \to \pi_n(F) \xrightarrow{i} \pi_n(E) \xrightarrow{p} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \ldots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

Note that starting from $\pi_1(E)$ we only get exact sequence of pointed sets; before that it's an exact sequence of groups. The long exact sequence follows from that of the pair (E, F), using the fact that $p : \pi_n(E, F) \to \pi_n(B)$ is an isomorphism.

The situation for (co)homology is not as simple as a long exact sequence, but instead *a lot of them*, put together in some systematic manner called the *spectral sequence*. This is the topic of the next lecture.

Now let us observe that it makes sense to talk about "the fiber" of a Hurewicz or Serre fibration.

Lemma 4. *Any two fibers of a Hurewicz (resp. Serre) fibration are homotopy equivalent (resp. weakly equivalent), provided that the base space B is path connected.*

Proof. See Problem 6 and 7 of Homework 1. □

Because of this, we often also write a fibration as $F \to E \to B$ when it is clear that B is path-connected.

Now, if $p : E \to B$ and $p' : E' \to B'$ are two Serre fibrations, and F, F' are the respective fibers over the basepoints. A mapping between the two Serre fibrations is a pair of mapping $f : E \to E', g : B \to B'$ such that the following diagram commute:

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\
B \arrow[r, "g"] & B'
\end{tikzcd}$$

One can check that this induces a mapping $h : F \to F'$ between the fibers.

Lemma 5. *If any two of the three maps in f, g, h are weak equivalences, then so is the third one.*

Proof. Observe that a mapping between two Serre fibrations induce a mapping between the two associated homotopy LES. Now apply five lemma using the fact that two of the maps are weak equivalences (so they induce iso everywhere) to conclude that the third one also induces iso everywhere. □

For a later discussion on model categories (in Lecture 13), we also include two results regarding the factorization of maps.

Lemma 6. *Any map $f : X \to Y$ can be factored into $X \xrightarrow{g} X' \xrightarrow{h} Y$, where g is a homotopy equivalence and h is a Hurewicz fibration.*

Proof. Let e_0 be the map $g \mapsto g(0)$. Consider the pullback of $f : X \to Y$ along e_0; this yields another space $Z = \{(x, \gamma) \mid x \in X, \gamma \in Y^I, \gamma(0) = f(x)\}$. One can check that $Z \xrightarrow{ev_1:(x,\gamma)\mapsto\gamma(1)} Y$ is a Hurewicz fibration, and that $Z \xrightarrow{(x,\gamma)\mapsto x} X$ is a homotopy equivalence, with the inverse s being attaching the constant path at x. Now let $X' = Z$, $g = s$, $h = ev_1$. □

Lemma 7. *Any map $f : X \to Y$ can be factored into $X \xrightarrow{g} X' \xrightarrow{h} Y$, where g is a cofibration and h is a homotopy equivalence.*

Proof. Let X' be the mapping cylinder of f, g the inclusion into the cylinder, and h the retract onto the other side of the cylinder. □

4 Spectral Sequences

Introducing spectral sequences.

4.1 Motivation

Suppose we have a fiber bundle $p : E \to B$ of CW complexes. Let $B^{(n)}$ be the n-skeleton. Consider the cellular chain complex $C_n^{Cell}(B) = H_n(B^{(n)}, B^{(n-1)}) = \bigoplus_{n\ cells} H_n(D^n, \partial D^n)$. Consider the pullback

$$\begin{tikzcd} E^{[n]} \arrow[r] \arrow[d] & E \arrow[d] \\ B^{(n)} \arrow[r] & B \end{tikzcd}$$

where the function $E^{[n]} \to B^{(n)}$ is again a fiber bundle. Our goal is to understand the (co)homology of E from that of B and F. Consider the long exact sequence, $\ldots \to H_n(E^{[n-1]}, E^{[n-2]}) \to H_n(E^{[n]}, E^{[n-2]}) \to H_n(E^{[n]}, E^{[n-1]}) \xrightarrow{d} H_{n-1}(E^{[n-1]}, E^{[n-2]}) \to \ldots$. If we understand the first and the third terms as well as the mapping d, then we might understand the second term. Continuing this process, we might understand $H_n(E^{[\infty]}, E^{[0]}) = H_n(E)$.

Next, consider the pullback square

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \arrow[r, "\Phi"] & B \end{tikzcd}$$

where the Φ map is the characteristic map. For the pullback fiber bundle, since the base space is contractible, one can show that E' is homotopy equivalent to $D^n \times F$, and then from there observe that $(E^{[n]}, E^{[n-1]})$ is relatively homeomorphic to $\coprod_{n\ cells\ of\ B} (D^n \times F, S^{n-1} \times F)$, where F is the fiber $p^{-1}(*)$. Thus $H_*(E^{[n]}, E^{[n-1]}) = \bigoplus_{n\ cells} H_*(F \times D^n, F \times S^{n-1}) = C_n^{cell}(B) \otimes H_*(F)$. If B has (trivial fundamental group?), the mapping $H_*(E^{[n]}, E^{[n-1]}) \to H_*(E^{[n-1]}, E^{[n-2]})$ becomes $C_n^{cell}(B) \otimes H_*(F) \to C_{n-1}^{cell}(B) \otimes H_*(F)$, which is $d^{cell} \otimes id$ (this is nontrivial). This indicates that we should look at $H_*(B; H_*(F))$ in order to obtain $H_*(E)$.

(Note that Hatcher uses this way, but the argument is not as nice as what we'll present below, e.g. that $d^{cell} \otimes id$ part is nontrivial to prove and invokes Hurewicz's theorem, which we want to prove using spectral sequences.)

4.2 Spectral Sequences: Definition

Definition 7 (Spectral Sequence)**.** *A spectral sequence consists of a sequence $(E_r, d_r : E_r \to E_r)$ (each is a chain complex) where $d_r^2 = 0$, E_r abelian groups, together with an isomorphism $E_{r+1} = \ker d_r / \operatorname{im} d_r$.*

One place spectral sequences come from is an exact couple.

Definition 8 (Exact Couple)**.** *An exact couple is a long exact sequence $\ldots \xrightarrow{h} D \xrightarrow{f} D \xrightarrow{g} \xrightarrow{E} \xrightarrow{h} D \xrightarrow{f} \ldots$, which we usually write as a triangle:*

$$\begin{tikzcd} D \arrow[rr, "f"] & & D \arrow[dl, "g"] \\ & E \arrow[ul, "h"] & \end{tikzcd}$$

For instance, $\ldots \to H_*(E^{[n-1]}) \to H_*(E^{[n]}) \to H_*(E^{[n]}, E^{[n-1]}) \to \ldots$. We can let $D = \bigoplus_n H_*(E^{[n]})$, $E = \bigoplus_n H_*(E^{[n]}, E^{[n-1]})$, and with the obvious mappings they fit into an exact couple.

Suppose (D, E, f, g, h) is an exact couple. The *derived exact couple* is constructed as follows. Connect two exact couple triangles side by side:

$$\begin{tikzcd} D \arrow[rr, "f"] & & D \arrow[dl, "g"] \arrow[rr, "f"] & & D \arrow[dl, "g"] \\ & E \arrow[ul, "h"] & & E \arrow[ul, "h"] & \end{tikzcd}$$

Then we can get $d = g \circ h : E \to E$. One can check that $d^2 = 0$. Let $E' = \ker d / \operatorname{im} d$, and $D' = \operatorname{im} f$. $f' : D' \to D'$ is now again given by f. $g' : D' \to E'$ is given by $g'(f(x)) = g(x)$. It's in $\ker d$ and thus in E'. Now, for two choices x, x' of x, their difference lies in $h(E)$, and thus its image is in $\operatorname{im} d$, i.e. the choice of x doesn't matter. Finally, $h' : E' \to D'$ is given by $h'(x) = h(\tilde{x})$ where $\tilde{x} \in \ker d$ represents x in E'. One can check this yields another derived couple. Finally, we get a spectral sequence from a derived couple by letting $E_1 = (E, d)$ and $E_r = (E'_{r-1}, d'_{r-1})$.

One place to get exact couples is from filtered chain complexes. Given a sequence of chain complexes $(C_*^{(1)}, C_*^{(2)}, \ldots)$. Let $D = H_*(C^{(n)})$, $E = H_*(C^{(n)}/C^{(n-1)})$, then this gives the long exact sequence of homology. For instance, we can take $C_r^{(n)} = C_r(E^{[n]})$.

Filtered chain complexes also come from double chain complexes. A double chain complex is a set $C_{ij}, i, j \geq 0$, where we have vertical differentials $d_v : C_{pq} \to C_{p(q-1)}$, and horizontal differentials $d_h : C_{pq} \to C_{(p-1)q}$, such that the obvious diagram commutes (with an additional -1). For instance, $C_*(X) \otimes C_*(Y)$ would yield a double chain complex by letting d_h and d_v be the differentials on X and Y respectively. From a double chain complex we get a total chain complex: $C_n = \displaystyle\bigoplus_{p+q=n} C_{pq}$, and $d : C_n \to C_{n-1}$ is $d_h + d_v$. One can check that with the additional -1 this yields $d^2 = 0$. There are two ways to filter this chain complex: one is $C_*^{(n)} = \displaystyle\bigoplus_{p \leq n} C_{pq}$, and another is $\displaystyle\bigoplus_{q \leq n} C_{pq}$.

5 More on Spectral Sequences

Again let's consider a total chain complex $C_n = \bigoplus_{p+q=n} C_{pq}$, and the associated filtered chain complexes $\mathrm{Filt}_n^k = \bigoplus_{p+q=n, p\leq k} C_{pq}$. The exact couple is $D = H_*(\mathrm{Filt}^k), E = H_*(\mathrm{Filt}^k/\mathrm{Filt}^{k-1})$. Then we see that $E = E_1$ is the homology of the kth column (note that the quotient is pretty much just the corresponding entry on the kth column). More precisely, E^1_{pq} is the pth homology of the qth column, and the $d : E \to E$ differential turns out to be transition differential of the columns that sends H_p to H_{p-1}.

In a general exact couple, there is just one E-term. When it comes from a double chain complex, of course E would have bi-grading: $E^r = \bigoplus_{p,q} E^r_{p,q}$. The mapping d_r then sends $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$, as one can check.

Let's get some examples. Consider the following double chain complex, where all the nonzero morphisms are isomorphism:

$$\begin{array}{cccc}
\mathbb{Z} \longleftarrow & \mathbb{Z} & 0 & 0 \\
 & \big\downarrow & & \\
0 & \mathbb{Z} \longleftarrow & \mathbb{Z} & 0 \\
 & & \big\downarrow & \\
0 & 0 & \mathbb{Z} \longleftarrow & \mathbb{Z}
\end{array}$$

We shall do the computation page by page. Since the E_1 terms compute the column homologies, we see that E_1 page looks like this:

$$\begin{array}{cccc}
\mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}
\end{array}$$

Now the E_2 page stays the same because the arrow isn't long enough yet. As for the d_3, which reaches from $\mathbb{Z}$ to $\mathbb{Z}$, it is an interesting exercise to see that it's actually an isomorphism.

Stable Terms Note that because everything is 0 outside the first quadrant, for every fixed (p, q), the term E^r_{pq} eventually stabilizes as $r \to \infty$, and that stable term we call it E^∞_{pq}.

Consider the total chain complex C_*, and the associated homology $H_n(C_*)$. It turns out that for each n, the $\{E^\infty_{pq}\}_{p+q=n}$ terms yield a filtration of the homology group, more precisely we have $0 \subseteq F_0 \subseteq \dots F_n \subseteq F_{n+1} = H_n(C_*)$ where $E^\infty_{pq} = F_{p+1}/F_p$.

5.1 Serre Spectral Sequence

Suppose we have a Serre fibration $F \to E \xrightarrow{p} B$, and suppose for the moment that B is simply connected. Then there is a spectral sequence where $E^2_{pq} = H_p(B; H_q(F))$, and it converges to $H_{p+q}(E)$, meaning that there is a filtration $H_n(E)$ whose associated graded group is $\bigoplus_{p+q=n} E^\infty_{pq}$.

Consider for example the fibration $\Omega S^{n+1} \to PS^{n+1} \xrightarrow{p} S^{n+1}$, where PX is the path space (and thus PS^{n+1} is contractible). We know the homology of PS^{n+1} and S^{n+1}, so this will give us the homology of ΩS^{n+1}. (Charles: I believe SSAT has this example. Yep, Example 1.5 on Page 9 of Chapter 1.) The fact that $(0, 0)$, which is $H_0(\Omega S^{n+1})$, stabilizes to $H_o(PS^{n+1}) = \mathbb{Z}$, tells us that ΩS^{n+1} is connected, thus $\pi_1 S^{n+1} = \pi_0(\Omega S^{n+1}) = 0$ for $n \geq 1$. Some more computations happens and we conclude that $H_j(\Omega S^{n+1}) = \mathbb{Z}$ if j is a multiple of n, and 0 otherwise.

Problem 2. *Suppose X is simply connected and $H_i(X) = 0$ for $0 \leq i < n+1$, what can you say about $H_*(\Omega X)$ for $* < 2n$?*

Answer See Lecture 8.

6 And More Spectral Sequences

6.1 The Extension Problem

Note that D^r term is the direct sum of images in $H_*(\mathrm{Filt}_r) \to H_*(\mathrm{Filt}_{k+r})$. For fixed n and sufficiently large $r \gg 0$, $H_n(\mathrm{Filt}_{k+r}) = H_n(C_*)$. At this point, $D^r \to D^r$ is the inclusion $\mathrm{im}(H_n(\mathrm{Filt}_k)) \subseteq \mathrm{im}(H_n(\mathrm{Filk}_{k+1})) \subseteq \ldots \subseteq H_n(C_*)$ and the E_r term is the quotient of the images. Then for sufficiently large r, we have a filtration $F_0 \subseteq \ldots \subseteq F_n \subseteq F_{n+1} = H_n(C_*)$ where $E^r_{pq} = F_p/F_{p+1}$, such that $p + q = n$. Then we have the filtration of $H_n(C_*)$, but we still need to solve for the actual $H_n(C_*)$. This is called solving the extension problem.

The chain complex we considered is the following:

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[d, "2"] & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[l, "2"] \arrow[d, "2"] & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & \mathbb{Z} \arrow[l, "2"] \arrow[d, "2"] & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z} & \mathbb{Z} \arrow[l, "0"] & \mathbb{Z} \arrow[l, "2"]
\end{tikzcd}$$

We'll do direct computation that looks kind of ugly. First do the computation with total chain complex: (Skip the long and boring computation.)

Another example is given by considering both the spectral sequence and the total chain complex to investigate the H_4 (filtration of three $\mathbb{Z}_2$s) and H_5 ($\mathbb{Z}_2$) of the total space.

Then we run the spectral sequence in a different filtration to solve the extension problem and observe that H_4 is actually $\mathbb{Z}_8$.

6.2 Local Systems

Definition 9. *Let X be a space. A local system on X consists of:*

1. *A set $\{A_x \mid x \in X\}$ of Abelian groups.*

2. *For every 1-simplex $\Delta[1] \xrightarrow{\gamma} X$ an isomorphism $\gamma_* : A_{\partial_1(\gamma)} \to A_{\partial_0(\gamma)}$.*

Such that for every 2-simplex $c : \Delta[2] \to X$, which has edges c_{01}, c_{02}, c_{12}, we have $(c_{02})_* = (c_{01})_* \circ (c_{12})_*$, and such that if γ is constant, $\gamma_* = id$.

Equivalently, consider the fundamental groupoid $\pi_{\leq 1}X$ of X, which is the category where objects are points in X, morphisms are homotopy classes of paths rel endpoints. Then a local system is a functor from $\pi_{\leq 1}X$ to the Abelian groups.

Suppose $X \to B$ is a Hurewicz fibration, then $b \mapsto H_n(p^{-1}(b))$ gives a local system. In fact this is also true for Serre fibration. (Use the fact that a weak equivalence $X \to Y$ induces isomorphism on all homology groups.)

Now suppose X is a space with local system A. Then define $C_n(X; A) = \bigoplus_{c:\Delta[n]\to X} A_{c_0}$ where c_0 is the image of the first vertex under c. Define a differential $d : C_n(X; A) \to C_{n-1}(X; A) = \sum_i (-1)^i d_i$, where i ranges over the index set for all the $(n-1)$ simplexes. Let $d_i c$ to be the composition of $\Delta[n-1] \to \Delta[n]$ and $c : \Delta[n] \to X$. (The rest is on the notes, we'll figure it out later.) This then defines homologies with local coefficients.

7 Snow Day

Class cancelled due to heavy snow. The self-reading is on the construction of Serre Spectral Sequences, with notes that I unfortunately cannot upload. This is a standard topic that requires some effort to build up correctly, so I'll just refer readers to standard literature (e.g. Hatcher) for reference.

8 More on SSS, and the Hurewicz Theorem

8.1 Mappings on the E2 page; Transgression

Consider a spectral sequence. There are two regular chain complexes in there, the left column and the bottom row. Thus there is a mapping from $H_*(\text{left column}) \to H_*(C)$, and there is another map $H_*(C) \to E^2_{*,0}$ (the bottom row) by modding out everything else. These are called *edge homomorphisms*.

Suppose now we have a Serre filtration such that $\pi_{\leq 1}B$ is trivial. Then the edge homomorphism is induced by the maps $F \to E$ and $E \to B$.

Consider an element on the bottom row on the E^2 page. Then the differentials are obstructions of such element $\in H_*(B)$ from coming from $H_*(E)$. Likewise, for an element on the left column, the kernel of $F \to E$ are those that are "hit" by some differential coming from below.

Now suppose $\pi_{\leq 1}B$ is trivial, $H_*(B) = 0$ for $* < n$, $H_*(F) = 0$ for $* < m$, then from the graph of the E^2 page one sees that there is a canonical mapping defined as $H_p(B) \to H_{p-1}(F)$ for $p < n+m+1$. This mapping is usually denoted as τ and is called the *transgression*.

How do we understand the transgression geometrically? To have all of the $d_r a = 0$ for $r < |a|-1$ is equivalent to say that a is in the image of $H_*(E, F)$ (under the connecting homomorphism δ?). Define $a \in H_p(B)$ to be *transgressive* if there is some $\tilde{a} \in H_p(E, F)$ such that it is the corresponding term in $H_*(E, F)$ by the mapping of pairs $(E, F) \to (B, *)$. Then the transgression of a is defined as $\delta(\tilde{a})$ where δ is the connecting homomorphism $H_n(E, F) \to H_{n-1}(F)$.

Theorem 8.1. *Consider the Serre Spectral Sequence $\Omega B \to PB \to B$ for $\pi_{\leq 1}(B) = *$, $\tilde{H}_p(B) = 0$ for $p < n$. The the transgression $\tau : H_p(B) \to H_{p-1}(\Omega B)$ is an isomorphism for $p < 2n$.*

8.2 The Hurewicz Theorem

Theorem 8.2 (The Hurewicz Theorem)**.** *If $\pi_{\leq 1}(B) = *$, and $\pi_i(B) = 0$ for $i < n$ for $n \geq 2$, then the Hurewicz homomorphism $\pi_n(B) \to H_n(B)$ is an isomorphism. In other words, the first nonvanishing homotopy group is isomorphic to the first nonvanishing homology.*

Theorem 8.3 (Poincare)**.** *If B is connected, then the map $\pi_1(B)_{ab} \to H_1(B)$ is an isomorphism.*

Proof. Skipped. □

Proof of the Hurewicz Theorem. Induction on n, starting with $n = 1$. (This is Poincare's Theorem.) Suppose $n \geq 2$. Of course, $\pi_i(B) = 0$ for $i < n$, which by induction implies $H_i(B) = 0$ for $i < n$. By the transgression theorem above, $H_i(B) = H_{i-1}(\Omega B)$ for $i < 2n$, which in particular tells us $H_n(B) = H_{n-1}(\Omega B)$ for $n \geq 2$. Now look at the mapping $H_n(PB, \Omega B) \to H_{n-1}(\Omega B)$ and $H_n(PB, \Omega B) \to H_n(B)$. But look at the corresponding homotopy maps $\pi_n(PB, \Omega B) \to \pi_{n-1}(\Omega B)$ and $\pi_n(PB, \Omega B) \to \pi_n(B)$. $H_n(PB, \Omega B) = H_{n-1}(\Omega B)$ is iso by what we said above, $H_{n-1}(\Omega B) = \pi_{n-1}(\Omega B)$ by induction, $\pi_n(PB, \Omega B) = \pi_{n-1}(\Omega B)$ by the contractibility of PB (by the relative homotopy sequence), $H_n(PB, \Omega B) = H_n(PB/\Omega B) = H_n(B)$ because and $\pi_n(PB, \Omega B) = \pi_n(B)$ by lemma 4.5 at here, and it follows that $H_n(B) = \pi_n(B)$. □

As a corollary, we can get $\pi_i(S^n) = 0$ for $i < n$, and $\pi_n(S^n) = \mathbb{Z}$. First realize that $\pi_1(S^n) = 0$ by Van-Kampen, then $\pi_2(S^n) = H_2(S^n) = 0$, etc. And then this pushes all the way to $\pi_n = H_n = \mathbb{Z}$.

Now another example. Let A be any Abelian group, then we can construct $M(A, n)$ by first taking a free resolution $0 \to F_2 \to F_1 \to A \to 0$ and the build the corresponding sequence of space $\bigvee S^n \to \bigvee S^n \to M(A, n)$. Then by Hurewicz, we get a space that has $\pi_i = 0$ for $i < n$ and A for $i = n$. (See the end of lecture 9 for the rest of this story.)

9 Relative Hurewicz Theorem

Theorem 9.1 (Relative Hurewicz Theorem)**.** *Suppose we have a pair of spaces (X, A), X is connected. If $\pi_i(X, A) = 0$ for $i < n$, then $H_i(X, A) = 0$ for $i < n$, and $\pi_n(X, A) \cong H_n(X, A)$ for $n \geq 3$.*

Recall that $\pi_n(X, A) = \pi_{n-1}(F)$ for some other space F, this is why we have $n \geq 3$.

Proof. Now given $f : A \to X$, then we can factor it into $A \xrightarrow{\cong} \tilde{A} \to X$ where the first map is a homotopy equivalence, and the second map is a Hurewicz fibration. Recall that $\tilde{A} = \{(\gamma, a) \in X^I \times A \mid \gamma(1) = a\}$, or equivalently, it is the pullback between f and $X^I \xrightarrow{\gamma(1)} A$.

Now let us look at the Serre Spectral Sequence of $F \to \tilde{A} \to X$. Then we see that $H_p(X; \underline{H}_q(F)) \Rightarrow H_{p+q}(\tilde{A}) = H_{p+q}(A)$.

Now note that $\pi_1(A) \to \pi_1(X) \to \pi_1(X, A)$. The third one is 0 by assumption, thus the mapping $\pi_1(A) \to \pi_1(X)$ is surjective. By the following fact, the entirity of $\pi_1(X)$ acts trivially on $H_*(F)$.

Exercise 1. *If $Y \to B$ is a Serre fibration, and an $\alpha \in \pi_1(B)$ is in the image of $\pi_1(Y)$, then α acts trivially on the homology of the fiber.*

By assumption, $\pi_i(X, A) = \pi_{i-1}(F) = 0$ for $i < n$. By the absolute Hurewicz Theorem, $H_i(F) = 0$, so $H_*(X; H_i(F)) = 0$ for $i < n-1$. Then by the homology long exact sequence, for $i < n-1$, $H_i(X) \cong H_i(X) \implies H_i(X, A) = 0$ for $i < n-1$, and $H_i(A) \to H_i(X)$ is epi for $n-1$, thus $H_i(X, A) = 0$ for $i < n$.

Since X is connected, $H_0(X; H_q(F)) = H_q(F)$. By considering the transgression (?), we have the exact sequence:

$$H_n(A) \to H_n(X) \to H_{n-1}(F) \to H_{n-1}(A) \to H_{n-1}(X) \to 0.$$

On the other hand, we also have the following exact sequence:

$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(A) \to \dots$$

If we can map the bottom sequence to the top sequence, then we can invoke the five lemma.

There is a mapping $A \cup CF \to \tilde{A} \cup CF \to X$, from which (modulo some magic) we get a map $(\tilde{A} \cup CF, A) \to (X, A)$. Then we have $H_{n-1}(F) \cong H_n(\Sigma F) \cong H_n(\tilde{A} \cup CF, A) \to H_n(X, A)$, which allows one to compare the two exact sequences above and conclude that $H_n(X, A) \cong H_{n-1}(F) \cong \pi_{n-1}(F)$ (by absolute Hurewicz Theorem), which is $\pi_n(X, A)$. $\square$

9.1 Application

Consider the fibration $S^n \xrightarrow{f} X \to X \cup_f D^{n+1}$. Then $H_n(X \cup D^{n+1}, X) = H_i(D^{n+1}, S^{n+1})$ (by excision), which is 0 for $i < n+1$ and $\mathbb{Z}$ for $i = n+1$. By the relative Hurewicz theorem, then we know that $\pi_i(X \cup D^{n+1}, X) = 0$ for $i < n+1$ and $\mathbb{Z}$ for $i = n+1$ ($n \geq 2$). In other words, $\pi_i(X) \to \pi_i(X \cup D^{n+1})$ is an iso for $i < n$ and epi for $i = n$. However, $\pi_{n+1}(X \cup D^{n+1}, X) \to \pi_n(X)$ is induced by f, so we know that $\pi_i(X \cup D^{n+1} \cong \pi_i(X)$ for $i < n$, and $\cong \pi_i(X)/f$ for $i < n$.

This leads to the action of "killing homotopy groups." More precisely, let $\bigvee S^{n+1} \to X \to X'$, so the first map is epi on π_{n+1}, then $\pi_i(X) \cong \pi_i(X')$ is iso for $i < n+1$, and $\pi_{n+1}(X') = 0$. Now repeat with X' by adding onto it the $n+2$ cells. Then we get a series of mappings $X \to P^n X$ such that $\pi_i(X) \cong \pi_i(P^n X)$ for $i \leq n$, and $\pi_i(P^n X) = 0$ for $i > n$, then we have a sequence $P^n X \to P^{n-1} X \to \dots$ which is called the **Postnikov Tower** of X.

Now recall from lecture 8 that we have, for any Abelian group A and any $n \geq 2$, some M_A such that $\pi_i M_A = A$ for $i = n$ and 0 for $i < n$. Now $P^n M_A$ has $\pi_n = A$ and $\pi_i = 0$ for $i \neq n$. This is called the **Eilenberg-MacLane space** $K(A, n)$. This construction makes us lose some intuitive connection with geometry (the Eilenberg-Maclane space is highly non-geometrical), but it does yield information about geometry.

As an example, consider the fibration $F \to S^n \to P^n S^n = K(\mathbb{Z}, n)$. Then from the long exact sequence, we can immediately conclude that $\pi_i(F) = \pi_i S^n$ for $i > n$, and 0 for $i \leq n$. Now by Hurewicz theorem, $H_i F = 0$ for $i < n+1$, thus $\pi_{n+1} S^n = \pi_{n+1}(F) \cong H_{n+1}(F)$. Keep going along this road, we can eventually compute things up to e.g. $\pi_{n+14} S^n$. (See Lecture 11 for more details.)

10 Representability and Puppe Sequences (Guest Lecturer: Hiro Takana)

10.1 Brown Representability

The big theorem to be deduced today is the following:

Theorem 10.1. *Fix any abelian group A. Then for any pointed CW complex (X, x_0), we have a natural isomorphism $[X, K(A, n)] \cong \tilde{H}^n(X, A)$.*

One says in this case that singular cohomology is *represented* by the Eilenberg-MacLane spaces. This is a special case of the more general Brown representation theorem. In particular, along with Whitehead's theorem this allows us to conclude that:

Corollary 2. *Any two CW complexes that are both $K(G, n)$ are homotopy equivalent.*

Remark 2. *If we take X to be a point, then we observe that $[X, K(A, n)] = * \implies \tilde{H}^*(X, A) = 0$.*

Remark 3. *If we consider $X = S^0$, then we see that $[S^0, K(A, n)] = \pi_0 K(A, n) = A$ if $n = 0$, and 0 if $n > 0$.*

The strategy for proving this theorem is to show that the object on the left side actually satisfies the Eilenberg-Steenrod axioms, then along with the fact that it behaves in the same way as singular cohomology on $H^0(*)$, we can conclude the theorem from the uniqueness theorem[1]. More concretely, let us fix an abelian group A. We shall show that there is a functor $K^n : \mathbf{CWpair^{op}} \to \mathbf{AbGrp}$ given as $(X, B) \to [(X, B), K(A, n)]$ that satisfies the Eilenberg-Steenrod axioms.

Of course, it would be comforting to know the following:

Remark 4. *For any space X, $[X, K(A, n)]$ is an abelian group.*

Proof. Observe that $[X, K(A, n)] = [X, \Omega^2 K(A, n-2)]$. Now, a mapping $f \in [X, \Omega Y]$ is given by $x \mapsto f_x : S^1 \to Y$, so the composition of loops yield a group structure, and the second Ω guarantees commutativity, just as we observed for higher homotopy groups. Or one can observe that $[X, \Omega^2 K(A, n-2)] = [\Sigma^2 X, K(A, n-2)]$ and use the abelian group structure on $[\Sigma^2 X, Y]$ as explained in May's Book[2]. $\square$

Now we need to check the E-S axioms. The additivity axiom is immediate from the natural isomorphism $[\bigvee_i X_i, K(A, n)] \cong \prod_i [X_i, K(A, n)]$. Homotopy and dimension axioms are trivial, and the excision axiom follows from Mayer-Vietoris property, which is again trivial for the functor that we specified. So it remains to check the exactness axiom, which we'll do by investigating the Puppe Sequence.

10.2 Puppe Sequences

Let us fix a CW pair $A \subseteq X$ (but really any HEP pair works), so there is a monomorphism $i : A \hookrightarrow X$. Now consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
* \arrow[r] & X/A \cong C(A) \cup_i X
\end{tikzcd}$$

where on the bottom right (which is the pushout) we attach the cone $C(A)$ along i to X, thereby collapsing A to a point and obtain X/A as the result. We can, however, continue this process by pushing out along the trivial map again:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X/A \cong C(A) \cup_i X \arrow[r] & (C(A) \cup_i X)/X \cong \Sigma A
\end{tikzcd}$$

The isomorphism on the bottom right is immediate if one draws a graph to convince oneself. Note that this step shows the difference between topological construction and the pure "algebraic" analogue, say for groups: if we do this for a group diagram, we would be gettin $(X/A)/X = *$ instead of ΣA. If we continue this process for a few more steps we will get the following:

[1] This book, Theorem 1.31.

[2] Here, page 58.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] & \\
* \arrow[r] & X/A \cong C(A) \cup_i X \arrow[r] \arrow[d] & (C(A) \cup_i X)/X \cong \Sigma A \arrow[r] \arrow[d] & * \arrow[d] \\
& * \arrow[r] & \Sigma X \arrow[r] & \Sigma(X/A)
\end{tikzcd}$$

Now we need an easy observation:

Lemma 8. $\Sigma X/A = \Sigma X/\Sigma A$.

Proof. Omitted. □

Then from the graph above we obtain the following exact sequence, known as the **Puppe sequence**:

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma X/\Sigma A \to \Sigma^2 A \to \dots$$

Moreover, any two consecutive maps of this sequence define a pushout.

Corollary 3. *For any space Z, the sequence of pointed sets*

$$[A, Z] \leftarrow [X, Z] \leftarrow [X/A, Z] \leftarrow \dots$$

is exact.

Corollary 4. *When $Z \cong K(A, n)$, for each CW pair (X, B), we have a long exact sequence of abelian groups.*

This completes the proof of the theorem. As a final remark, we note that the same construction also yields an exact sequence for fibration, as can be read off from the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \Omega E \arrow[r] \arrow[d] & \Omega B \arrow[r] \arrow[d] & * \arrow[d] \\
\dots \arrow[r] & * \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[d] \\
& & * \arrow[r] & B
\end{tikzcd}$$

11 Cohomological SSS (Guest Lecturer: Xiaolin Shi)

Today we will introduce the cohomological analogue of Serre Spectral Sequence and use it to do some computations.

Definition 10 (Cohomological Serre Spectral Sequence). *Given a Serre fibration $F \to E \to B$, we have a spectral sequence $E_2^{p,q} = H^p(B; H^q(F)) \Rightarrow H^{p+q}(E)$, with differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. The infinity page again has a filtration: $H^{p+q}(E) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^{p+q}$, where $E_\infty^{p,q} = F^p/F^{p+1}$.*

The ring structure that cohomology carries extends naturally to a ring structure on the CSSS.

Proposition 3. *There exists a multiplicative structure $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ on each page, which we denote by $(x, y) \mapsto xy$. The differential d_r acts as a derivation on this structure, i.e. $d_r(xy) = xd_r(y) + (-1)^{|x|}d_r(x)y$.*

11.1 An Appetizer

Let us consider the Eilenberg-MacLane spaces. It is clear that $K(\mathbb{Z}, 1) = S^1$. (For a quick proof that the higher homotopy groups vanish, consider $S^n \to S^1$ for $n > 1$; since S^n is simply connected, the map lifts to $S^n \to \mathbb{R}$, and since $\mathbb{R}$ is contractible this mapping must be nulhomotopic, and this property descends down to $S^n \to S^1$ via projection.) On the other hand, from the cohomology ring structure we know that $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$. But suppose we don't already know this, can we deduce this directly from the definition of $K(\mathbb{Z}, 2)$? Yes.

Let us consider the pathspace fibration $\Omega K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$. As we noted before, $\Omega K(\mathbb{Z}, 2) = K(\mathbb{Z}, 1) = S^1$, and $PK(\mathbb{Z}, 2)$ is contractible. This allows us to write out the spectral sequence $E_2^{p,q} = H^p(K(\mathbb{Z}, 2); H^q(S^1)) \Rightarrow H^{p+q}(*)$, which is as follows:

$$\begin{tikzcd}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & H^0 \arrow[drr] & H^1 & H^2 & H^3 & H^4 \\
0 & H^0 & H^1 & H^2 & H^3 & H^4 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

Everything above the second line is zero, and the first two rows are also rather straightforward (H^k denotes the kth cohomology of $K(\mathbb{Z}, 2)$, of course). The only nontrivial differential on the E_2 page is then the arrows from H^k to H^{k+2} as indicated in the graph. Next, observe that E_3 and later pages have no nontrivial differential, so the SSS collapses after the E_2 page. Thus on the E_∞ page, we have, on the left-bottom corner:

$$\begin{array}{c|ccccc}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & H^0 & ? & ? & ? & ? \\
0 & H^0 & H^1 & ? & ? & ? \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

Compare this with the diagram that we should get from the cohomology of a point:

$$
\begin{array}{c|ccccc}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

we immediate conclude that $H^0 = \mathbb{Z}$ and $H^1 = 0$. Furthermore, since the arrow indicated on the first graph kills the H^0 at $(0,1)$, we know that that particular arrow must be injective; and since the H^2 at $(2,0)$ also dies, we know it is also surjective and thus bijective. In this way, we immediately conclude that $H^{2n}(K(\mathbb{Z},2)) = 0$, and $H^{2n+1}(K(\mathbb{Z},2)) = 0$. It remains to identify the ring structure.

Now we have the following diagram as E_2:

$$
\begin{tikzcd}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathbb{Z}_{(a)} \arrow[drr] & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\
0 & \mathbb{Z}_{(1)} & 0 & \mathbb{Z}_{(x_2)} & 0 & \mathbb{Z}_{(x_4)} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

where some of the generators have been named and marked at the corner of the corresponding $\mathbb{Z}$ entries. Now, because a generates H^0 at $(0,1)$, one can do direct computation and see that ax_2 generates $(2,1)$, and similarly ax_4 for $(4,1)$, etc. Choose the sign of a and x_2 appropriately so that $d(a) = x_2$, then since $d(x_2) = 0$, we have $d(ax_2) = d(a)x_2 + (-1)^1 ad(x_2) = x_2^2 = x_4$, and similarly $x_6 = x_2^3$, etc. This allows us to conclude that $H^*(K(\mathbb{Z},2)) = \mathbb{Z}[x_2]$. Finally, to conclude that this uniquely identifies $K(\mathbb{Z},2)$, we can either appeal to the result from last lecture, or observe the fibration $S^1 \to S^\infty \to \mathbb{CP}^\infty$, induced as the limit of $S^1 \to S^{2n+1} \to \mathbb{CP}^n$, and investigate the homotopy long exact sequence associated with it (then apply Whitehead).

11.2 The Main Course

Theorem 11.1. $\pi_4(S^3) = \mathbb{Z}_2$.

This is the first stabilized term π_1^S of the stable homotopy groups.

Proof. We start by considering the fibration $X \to S^3 \to K(\mathbb{Z},3)$, where X is the homotopy fiber. As we mentioned before, this effectively "kills $\pi_3(S^3)$" because, as one can observe from the homotopy long exact sequence, we have $\pi_i(X) = \pi_i(S^3)$ for $i \geq 4$ and $\pi_i(X) = 0$ for $i \leq 3$. However, this fibration itself is a bit awkward to use, so we "back up a step" by looking at the Puppe sequence: $\ldots \to \Omega S^3 \to \Omega K(\mathbb{Z},3) \to X \to S^3 \to K(\mathbb{Z},3)$. We choose the fibration $\Omega K(\mathbb{Z},3) \to X \to S^3$ to apply CSSS on. Then we get the E_2 page as follows:

$$
\begin{tikzcd}
6 & \mathbb{Z}_{(a^3)} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & \mathbb{Z}_{(a^2)} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z}_{(a)} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z}_{(1)} & 0 & 0 & \mathbb{Z}_{(x)} & 0 & 0 & 0 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

where the nontrivial differentials are hown. Let generators of the 0th column be named $(1, a, a^2, \ldots)$ as shown in the graph, and the bottom one on the 3rd column be (x), then we know that the 3rd column's generators are $(x, ax, a^2x, \ldots)$.

Next note that the space X is 3-connected, so we can invoke the Hurewicz theorem to obtain that $\tilde{H}_i(X) = 0 \implies \tilde{H}^i(X) = 0$ for $i \leq 3$, and from which we immediately obtain that $d_3(a) = x$, as the differential needs to kill both $(0, 2)$ and $(3, 0)$. In this manner, we can figure out the other differentials using the derivation property, e.g. $d(a^n) = na^{n-1}d(a)$ for $|a| \equiv 0 \pmod 2$, and from this we can recover the E_∞ page and see that $H^{2i+1}(X) = \mathbb{Z}/i$ for $i \geq 2$. Then by the universal coefficient theorem, we have $H_{2i}(X) = \mathbb{Z}/i$ for $i \geq 2$, and thus $H_4(X) = \mathbb{Z}/2 = \pi_4(X)$ by Hurewicz. Since $\pi_4(X) = \pi_4(S^3)$, we have completed the proof.

$\square$

12 A Glance at Model Categories, Part 1 (Guest Lecturer: Emily Riehl)

Preliminary remarks: if A is a cell complex, $X \to Y$ is a weak homotopy equivalence, then $[A, X] \cong [A, Y]$. Also we have Whitehead's theorem: $X \cong Y$ between cell complex is a homotopy equivalence if and only if it's a weak homotopy equivalence.

We'll generalize this in an axiomatic framework for abstract homotopy theory, i.e. model category theory. This framework can also be used to do a few other things:

- One can use this to prove the equivalence of different homotopy theories (e.g. the case for simplicial sets);
- One can use this to get derived functor systematically (e.g. homotopy (co)limits).
- One can use this to present an $(\infty, 1)$-category.

12.1 Definition of Model Categories

Definition 11 (DHKS[3]). *A* ***homotopical category*** *is a category M with a class of maps W (the class of equivalences, containing all isos) that satisfies either of the following:*

- *2-of-3 property If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and two of $\{f, g, g \circ f\}$ are in W, then all of them are.*
- *2-of-6 property If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} N$ and $h \circ g, g \circ f \in W$, then all of $\{f, g, h, h \circ g, g \circ f, h \circ g \circ f\}$ are in W.*

(the second one is stronger than the first one, but in case for homotopical category they are equivalent.)

Definition 12 (Gabriel-Zisman). *A homotopical category has a* ***hopotopy category*** *$HoM = M[W^{-1}]$ with a localization functor $M \xrightarrow{\gamma} HoM$ (details omitted), given as follows:*

- *The objects of HoM are just objects of M.*
- *The morphism $x \to y$ is a finite zig-zag $x \leftarrow z_1 \to z_2 \ldots \to y$, up to equivalence described as the following:*

  $x \xleftarrow{f} z \xrightarrow{f} x = id_x$, *and*

  $x \xleftarrow{f} z \xleftarrow{g} y = x \xleftarrow{g \circ f} y$.

  If (M, W) can be expanded to a model category then we can define HoM in a nicer way.

Definition 13 (Quillen / Joyal-Tierney). *A* ***model structure*** *on a homotopical cateogry with all small limits and colimits (M, W) consists of a class $\mathcal{C}$ of cofibrations and a class $\mathcal{F}$ of fibrations, so that $(\mathcal{C} \cap W, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap W)$ are weak factorization systems (defined below).*

Definition 14. *A* ***weak factorization system*** *of a category M consists of two classes of maps (L, R) such that:*

- *Factorization Any $f \in M$ can be factored into $r \circ l$, where $l \in L$ and $r \in R$.*
- *Lifting If $l \in L, r \in R$, then the following diagram lifts:*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "l"'] & Y \arrow[d, "r"] \\ X' \arrow[r, "v"'] \arrow[ur, "\exists w"] & Y' \end{tikzcd}$$

  Note that this generalizes both HEP and HLP.

- *Closure $L = \{l \mid l \nearrow r \ \forall r \in R\}$, $R = \{r \mid l \nearrow r \ \forall l \in L\}$, where $\nearrow$ refers to the lifting diagram above (note this yields a Galois connection).*

(Reference: Homotopy theories and model categories. W. G. Dwyer and J. Spalinski.)

Example 4. *On the category* ***Set***, *$(mono, epi)$ is a weak factorization system. The factorization is the cograph factorization $X \to X \coprod Y \to Y$, and the lifting property and the closure property can be checked manually.*

[3] *Homotopy Limit Functors on Model Categories and Homotopical Categories.*

Lemma 9. *If Z is the class of maps with the left lifting property (i.e. being the L in the definition above) with respect to some class R, then Z is closed under coproduct, pushout, retract, transfinite composition and contains the isos. (The dual version for right lifting property also holds.)*

Proof. We prove the case for retract. Recall that f is a retract of g if there exists a diagram of the following:

$$\begin{tikzcd} A \arrow[r, "s_1"] \arrow[d, "f"] & X \arrow[r, "r_1"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B \end{tikzcd}$$

such that $r_1 \circ s_1 = id_A, r_2 \circ s_2 = id_B$. Then the rest of the proof is trivial. (...) $\square$

Example 5. *Some examples of model structures are as follows.*

- *(**Top**, homotopy equivalences, closed Hurewicz cofibrations, fibrations)*
- *(**Top**, weak homotopy equivalences, retracts of relative cell complexes, Serre fibrations)*
- *(Chain complexes over R-module, homotopy equivalences, Hurewicz cofibrations, Hurewicz fibrations)*
- *(Chain complexes over R-module, quasi-isomorphisms, retracts of relative cell complexes, Serre fibrations)*

Note that the third and the fourth ones are the respective analogies of the first and the second ones. (This hints at the fact that homological algebra is a special case of homotopical algebra modulo the language of model structures.)

Remark 5. *A few remarks to be made before the end of lecture.*

1. *The model axioms are self-dual.*
2. *If M is a model category, and X is an object in M, then X/M and M/X (the "slice categories") are model categories with a forgetful functor $X/M \to X$ that creates weak equivalences, cofibrations and fibrations.*

13 A Glance at Model Categories, Part 2 (Guest Lecturer: Emily Riehl)

Recall from last lecture the definition of model categories and weak factorization systems. As explained last time, the factorization axiom guarantees that any map in the cateogry can be factored into the composition of a trivial cofibration $\in W \cap \mathcal{C}$ and a fibration $\in \mathcal{F}$, and can be factored into a cofibration followed by a trivial fibration.

Another quick remark: given the factorization and the lifting axioms of the weak factorization system, the closure property is equivalent to the "retract closure" property that both L and R are closed under all retracts. This can be proved using a "retract argument" c.f. this note.

Today we consider the second example listed in last lecture, (**Top**, weak homotopy equivalences, $\mathcal{C}$ = retracts of relative cell complexes, $\mathcal{F}$ = Serre fibrations). This construction is due to Quillen. More explicitly, $(\mathcal{C}, \mathcal{F} \cap W)$ is the system where $\mathcal{F} \cap W$ are maps having the right lifting property against $S^{n-1} \to D^n$, and $\mathcal{C}$ are retracts of relative cell complexes built from $S^{n-1} \to D^n$. Dually, $(\mathcal{C} \cap W, \mathcal{F})$ is the system where $\mathcal{F}$ are maps having the RLP against $D^n \to D^n \times I$ and $\mathcal{C} \cap W$ are retracts of rel. cell complexes built from $D^n \to D^n \times I$.

Today's object is to prove the following two theorems at the full generality of model categories.

Theorem 13.1. *If A is a cell complex, f is a weak equivalence $X \to Y$, then $[A, X] \xrightarrow{f_*} [A, Y]$ is an isomorphism.*

Theorem 13.2 (Whitehead)**.** *If X, Y are cell complexes, then $X \xrightarrow{f} Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

13.1 (Co)fibrant Objects

Cell complexes are known as the fibrant-cofibrant (sometimes known as bifibrant) objects in the model category given above.

Definition 15. *A is a* ***cofibrant object*** *if and only if the unique mapping $\emptyset \to A$ from the initial object to A is in $\mathcal{C}$. Dually, A is a* ***fibrant object*** *if and only if the unique mapping $A \to *$ from A to the final object is in $\mathcal{F}$. A is called* ***fibrant-cofibrant*** *(or* ***bifibrant****) if both conditions are met.*

Example 6. *In the model category above, all objects are fibrant (this is an easy exercise), and the cell complexes are cofibrant.*

13.2 Homotopy in Model Categories

Any model category is equipped with a natural notion of homotopy such that

1. The whitehead theorem is true;
2. There exists well-defined homotopy classes of maps;
3. The category HoM is equivalent to the category M', where the objects are bifibrant objects of M and the morphisms are homotopy classes of maps.

Definition 16. *A* ***cylinder object*** *$cyl(A)$ for $A \in M$ is an object such that the following factorization exists:*

$$\begin{tikzcd} A \coprod A \arrow[rr, "id \coprod id"] \arrow[dr, "f=(i_0,i_1)"'] & & A \\ & cyl(A) \arrow[ur, "g"'] & \end{tikzcd}$$

where g is a weak equivalence. The object is called ***good*** *if f is a cofibration, and* ***very good*** *if additionally g is a trivial fibration.*

Note that in a model category we are always guaranteed to have a very good cylinder object. Also it's worth noting that cylinder objects are not uniquely defined.

Proposition 4. *Given a good cylinder object $cyl(A)$ with A a cofibrant, i_0 and i_1 are trivial cofibrations.*

Proof. Consider the following diagram, where the square is the defining pushout for coproduct:

$$
\begin{tikzcd}
\emptyset \arrow[r, "cofib"] \arrow[d, "cofib"] & A \arrow[d] \arrow[ddr, "i_1"] \arrow[dddrr, bend left, "id"] & & \\
A \arrow[r] \arrow[drr, "i_0"] \arrow[ddrrr, bend right, "id"'] & A\coprod A \arrow[dr, "cofib"] & & \\
& & cyl(A) \arrow[dr] & \\
& & & A
\end{tikzcd}
$$

We have marked in the diagram the cofibrations that have been given by the assumptions. By the pushout property of cofibrations, the two maps into $A\coprod A$ are cofibrations, then since cofibrations are closed under composition, i_0 and i_1 are cofibrations. By definition, $cyl(A) \to A$ is a weak equivalence, and id, being an isomorphism, also is a weak equivalence, so from the 2-of-3 property we conclude that i_0 and i_1 are weak equivalences. □

Definition 17. *Given two mappings $f, g : A \to X$, a* ***left homotopy*** *between f and g is a mapping H such that the following diagram commutes:*

$$
\begin{tikzcd}
A \arrow[dr, "i_0"] \arrow[drr, bend left, "f"] & & \\
& cyl(A) \arrow[r, "H"] & X \\
A \arrow[ur, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
$$

We denote this by $f \simeq^l g$.

Note that the "left" comes from the fact that model categories are self-dual. In particular, there is a dual notion of **path objects** and the associated notion of **right homotopy**.

Remark 6. *If $f \simeq^l g : A \to X$, then there exists a good left homotopy (where the cylinder object is good). In addition, if X is fibrant, then there exists a very good left homotopy.*

Proof. Just consider the following diagram, where the factorization through $cyl_{good}(A)$ is granted by the factorization axiom.

$$
\begin{tikzcd}
A\coprod A \arrow[rr] \arrow[dr, "\text{cofib}"'] & & cyl(A) \arrow[r, "H"] & X \\
& cyl_{good}(A) \arrow[ur, "\text{trivial cofib}"'] & &
\end{tikzcd}
$$

Now suppose X is fibrant. Choose a good left homotopy $H : cyl_{good}(A) \to X$. Now consider the mapping $g : cyl_{good}(A) \to A$, which is guaranteed to be a weak equivalence by the cylinder object definition. By the factorization axiom, we can factor g into $g_2 \circ g_1$, where g_2 is a trivial fibration and g_1 is a cofibration. By 2-of-3 property, this means that g_1 is also trivial. We write this as $cyl_{good}(A) \xrightarrow{g_1} cyl_{vg}(A) \xrightarrow{g_2} A$. Now consider the following diagram:

$$
\begin{tikzcd}
cyl_{good}(A) \arrow[r, "H"] \arrow[d, "g_1"] & X \arrow[d] \\
cyl_{vg}(A) \arrow[r] \arrow[ur, dotted, "H'"] & *
\end{tikzcd}
$$

By assumption, $X \to *$ is fibration and g_1 is trivial cofibration, so by the lifting axiom, the mapping H' exists, and one sees that $cyl_{vg}(A)$ yields a very good homotopy.

□

Proposition 5. *If A is cofibrant, then $\simeq^l$ defines an equivalence relation on $Hom(A, X)$.*

Note that regardless of whether A is cofibrant or not (?), we can always define $[A, X] = Hom(A, X)/\simeq^l$.

Proof. We skip the proof for reflexivity and symmetry. (Just choose the cylinder object judiciously.) Let us prove transitivity. Suppose we have two homotopies, given by the following diagrams:

$$
\begin{tikzcd}
A \arrow[dr, "i_0"] \arrow[drr, bend left, "f"] & & \\
& cyl(A) \arrow[r, "H"] & X \\
A \arrow[ur, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[dr, "i'_0"] \arrow[drr, bend left, "g"] & & \\
& cyl'(A) \arrow[r, "H'"] & X \\
A \arrow[ur, "i'_1"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

In order to construct the necessary cylinder object, consider the pushout of i_1 and i'_0 as follows:

$$
\begin{tikzcd}
A \arrow[r, "i'_0"] \arrow[d, "i_1"] & cyl'(A) \arrow[d, "\tilde{i_1}"] \arrow[ddr, bend left, "H'"] & \\
cyl(A) \arrow[r, "\tilde{i'_0}"] \arrow[drr, bend right, "H"'] & cyl''(A) \arrow[dr, "H''"] & \\
& & X
\end{tikzcd}
$$

Since $H \circ i_1 = H' \circ i'_0 = g$, we have the outer commutative square, and by the UMP of pushout we have the induced mapping $H'' : cyl''(A) \to X$. Then we have the following diagram:

$$
\begin{tikzcd}
A \arrow[dr, "\tilde{i'_0} \circ i_0"] \arrow[drr, bend left, "f"] & & \\
& cyl''(A) \arrow[r, "H''"] & X \\
A \arrow[ur, "\tilde{i_1} \circ i'_1"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

But we still need to verify that $cyl''(A)$ is actually a cylinder object. This can be checked via the following diagram:

$$
\begin{tikzcd}
A \arrow[r, "i'_0"] \arrow[d, "i_1"] & cyl'(A) \arrow[d, "\tilde{i_1}"] \arrow[ddr, bend left] & \\
cyl(A) \arrow[r, "\tilde{i'_0}"] \arrow[drr, bend right] & cyl''(A) \arrow[dr] & \\
& & A
\end{tikzcd}
$$

Since pushout of trivial cofibrations are again trivial cofibrations, $\tilde{i_1}$ and $\tilde{i'_0}$ are trivial (because A is cofibrant). On the other hand, $cyl(A) \to A$ and $cyl'(A) \to A$ are guaranteed to be trivial, so by 2-of-3 property we can conclude that $cyl''(A) \to A$ is trivial.

$\square$

Theorem 13.3. *If A is cofibrant, $X \xrightarrow{p} Y$ is trivial fibration, then $[A, X] \xrightarrow{p_*} [A, Y]$ is an isomorphism.*

Note that this is the model categoric statement of Theorem 13.1.

Proof. First we would like to check that p_* is well defined. Suppose we have $f \simeq^l g$, then by the following graph, we directly have $pf \simeq^l pg$.

$$
\begin{tikzcd}
A \arrow[dr, "i_0"] \arrow[drr, bend left, "f"] & & & \\
& cyl(A) \arrow[r, "H"] & X \arrow[r, "p"] & Y \\
A \arrow[ur, "i_1"] \arrow[urr, bend right, "g"'] & & &
\end{tikzcd}
$$

Now we want to show surjectivity. Consider the following commutative square on the left, where $k : A \to Y$ is some chosen element of $[A, Y]$:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "k"] \arrow[ur, dotted, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \coprod A \arrow[r, "f \coprod g"] \arrow[d] & X \arrow[d, "p"] \\ cyl(A) \arrow[r, "H"] \arrow[ur, dotted, "H'"] & Y \end{tikzcd}$$

As we mentioned above, $\emptyset \to A$ is in $\mathcal{C}$ since A is cofibrant, and $p \in \mathcal{F} \cap W$ by assumption, then the mapping f exists by the lifting axiom, and thus we have surjectivity. Now for injectivity: suppose that we have $pf \simeq^l pg$, then consider the right diagram above, where $A \coprod A \to cyl(A)$ is a cofibration as we have shown above, so H' exists, and thus $f \simeq^l g$. □

Proposition 6. *If X is fibrant, $A' \to h$ is a mapping, $f, g : A \to X$, then $f \simeq^l g \implies fh \simeq^l gh$.*

Proof. As we mentioned before, we can choose a very good left homotopy $A \coprod A \to cyl_{vg}(A) \xrightarrow{k} X$ for $f \simeq^l g$. On the other hand, consider the following diagram:

$$\begin{tikzcd} A' \coprod A' \arrow[r, "h \coprod h"] \arrow[d] & A \coprod A \arrow[r] & cyl_{vg}(A) \arrow[d] \\ cyl_{good}(A') \arrow[r] \arrow[urr, dotted, "H'"] & A' \arrow[r, "h"] & A \end{tikzcd}$$

where the unmarked maps are canonical. It is clear that the right side map is a trivial fibration, and the left side map is a cofibration, and thus H' exists, and one sees that $cyl_{good}(A') \xrightarrow{k \circ H'} X$ is a left homotopy between fh and gh. □

Proposition 7. *Suppose $f, g : A \to X$. If A is cofibrant then $f \simeq^l g \implies f \simeq^r g$, and if X is fibrant then $f \simeq^r g \implies f \simeq^l g$.*

Proof. Omitted. Some clever lifting argument. □

Theorem 13.4. *Let $f : A \to X$ where A and X are bifibrant objects. Then f is a weak equivalence if and only if f is a homotopy equivalence (i.e. there exists $g : X \to A$ such that $f \circ g \simeq 1_X, g \circ f \sim 1_A$[4]).*

Of course, this is the model categoric version of Whitehead's theorem, i.e. Theorem 13.2.

Proof. We will only prove the only if direction; for the other one, see Dwyer and Spalinski's notes (Lemma 4.24). Suppose f is a weak equivalence. Then by 2-of-3 property, we can factor it into a trivial cofibration followed by a trivial fibration. Suppose this is written as $f : A \xrightarrow{q} C \xrightarrow{p} X$. Now consider the following diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ X \arrow[r, "id"] \arrow[ur, dotted, "s"] & X \end{tikzcd}$$

$\emptyset \to X$ is cofibration because X is cofibrant, and $C \to X$ is a trivial fibration, so s exists, and clearly $p \circ s = id_X$. Next observe that by the theorem we proved above, we know that $[C, C] \xrightarrow{p_*} [C, X]$ is an isomorphism, so in particular $p_*([p \circ s]) = [p \circ s \circ p] = [p] = p_*([1_C])$, so we know that $p \circ s \sim 1_C$. Dually, this also gives us an inverse r to q such that $r \circ q = 1_A$ and $q \circ r = 1_C$. Together this yields a two-sided homotopy inverse for f.

□

[4]Since $\simeq^l$ and $\simeq^r$ are the same in this case, we are justified to just write $\simeq$. $[A, X]$ in this case is often written as $\pi(A, X)$.

14 Serre Classes

Welcome back, Mike. Okay now recall the Hurewicz theorem and the computation that we had in lecture 11. We learned that $H_4(X) = \mathbb{Z}_2 = \pi_4(S^3)$ for $X \to S^3 \to K(\mathbb{Z}, 3)$. Suppose we then use the Eilenberg-Maclane fibration $X_5 \to X \to K(\mathbb{Z}/2, 4)$, back up a step along the Puppe sequence and consider the sequence $K(\mathbb{Z}/2, 3) \to X_5 \to X$ (i.e. its Spectral Sequence):

$$
\begin{array}{c|ccccccccccc}
3 & \mathbb{Z}/2 & & & & & & & & & & \\
2 & 0 & & & & & & & & & & \\
1 & 0 & & & & & & & & & & \\
0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/3 & 0 & \mathbb{Z}/4 & 0 & \mathbb{Z}/5 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
$$

We don't know much about the 0th column (except the part revealed by Hurewicz), but it looks like that we have a bunch of 2-groups. The 0th row we have already calculated, on the other hand. Then we see that $\mathbb{Z}_3$ can't be effectively killed when it reaches these groups, nor do the following $\mathbb{Z}/5, \mathbb{Z}/7$, etc on the 0th row. Moreover, until the 6th page for $\mathbb{Z}/3$ (and 8th page for $\mathbb{Z}/4$, etc.), the calculation on earlier pages will not affect $\mathbb{Z}/3$ at all. This suggests at the fact that we should be able to simplify these calculations significantly by only looking at some particular groups "locally", and this is where the idea of Serre classes kick in.

14.1 Serre Classes

Definition 18. *A **Serre Class** of abelian groups is a class $\mathcal{C}$ of abelian groups such that if $0 \to A \to B \to C \to 0$ is an exact sequence then $B \in \mathcal{C}$ iff $A, C \in \mathcal{C}$.*

Example 7. *Some Serre classes are listed below:*

1. *The class of torsion abelian groups.*
2. *For some fixed p, the class of p-torsion abelian groups.*
3. *Let S be the set of primes, the class of A such that $\forall a \in A$, $\exists n = p_1^{e_1} \dots p_n e^n, p_i \in S, na = 0$.*
4. *The class of finitely generated abelian groups.*

Intuitively, the Serre classes are the elements that we hope to ignore in our computation.

Definition 19. *A map $A \xrightarrow{f} B$ is an epi mod $\mathcal{C}$ if coker $f \in \mathcal{C}$, is mono mod $\mathcal{C}$ if $\ker f \in \mathcal{C}$, and is iso mod $\mathcal{C}$ if both hold.*

Homological algebra extends naturally to the mod $\mathcal{C}$ version.

Exercise 2. *Prove that the mod $\mathcal{C}$ version of the five lemma holds.*

More systematically speaking, there is a universal functor $\mathcal{F} : \mathcal{A} \to \mathcal{A}_{\mathcal{C}}$ (the localization functor, which makes $A \to B$ an isomorphism if $A/B \in \mathcal{C}$) for an abelian category $\mathcal{A}$ and a collection of objects $\mathcal{C}$ (meeting certain closure conditions), such that (epi, mono, iso) mod $\mathcal{C}$ translates to (epi, mono, iso) in $\mathcal{A}_{\mathcal{C}}$.

Example 8. *Let $\mathcal{A}$ be abelian groups, and $\mathcal{C}$ be the torsion abelian groups. Then $A \to B$ is (epi, mono, iso) mod $\mathcal{C}$ if and only if $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is (epi, mono, iso). In this case, $\otimes\mathbb{Q}$ is the universal functor $\mathcal{F}$.*

Proposition 8. *If $\mathcal{C}$ is a Serre class, $A \in \mathcal{C}$, B is finitely generated, then $A \otimes B$, $Tor(A, B) \in \mathcal{C}$.*

Proof. $A \in \mathcal{C} \implies A \oplus A \in \mathcal{C}$ because $0 \to A \to A \oplus A \to A \to 0$. Thus $A^n \in \mathcal{C}$, i.e. $\mathbb{Z}^n \otimes A \in \mathcal{C}$. Now consider the exact sequence $0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to B \to 0$, then we have the sequence $0 \to \mathrm{Tor}(A, B) \to A^m \to A^n \to A \otimes B \to 0$, from which we can conclude that $\mathrm{Tor}(A, B), A \otimes B \in \mathcal{C}$. $\square$

Corollary 5. *If H_iX is finitely generated for each i, and $A \in \mathcal{C}$, then for each i, $H_i(X;A) \in \mathcal{C}$.*

Proposition 9. *For $A \hookrightarrow X$, if $H_i(X,A) \in \mathcal{C}$, then $H_iA \to H_iX$ is iso mod $\mathcal{C}$.*

Proof. Break up the long exact sequence of homology. □

Proposition 10. *Suppose $F \to E \to B$ is a Serre fibration such that $\pi_0 B = 0$, $\pi_1 B$ is abelian acting trivially on $\pi_n F$. If $H_iF \in \mathcal{C}$ for $i > 0$, $H_iE \in \mathcal{C}$ for $i > 0$, then $H_iB \in \mathcal{C}$ for $i > 0$, under some restrictive condition explained below.*

Proof. Consider the E^2 page mod $\mathcal{C}$.

$$\begin{array}{c|ccccc}
2 & 0 & ? & ? & ? & ? \\
1 & 0 & ? & ? & ? & ? \\
0 & \mathbb{Z} & ? & ? & ? & ? \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

From the assumption we know that the 0th column is all $0_{\mathcal{C}}$ ($x \in \mathcal{C}$, $x = 0$ mod $\mathcal{C}$, $x \in 0_{\mathcal{C}}$ are the same thing.). Naturally the next step is to try to clear the rest of the page. Let's first see a general attack strategy that, though unfortunately won't work in this case, gives us some interesting insights.

First consider the tail of the homological LHE: $\ldots \to H_1E \to H_1B \to 0$. This directly yields $H_1B \in \mathcal{C}$. Now look at $(1,1)$: It's $H_1(B;H_1F)$. For the sake of argument, suppose we also know that $H_i(B)$ are finitely generated, then we can conclude that $H_i(B;H_iF) \in \mathcal{C}$ by an universal coefficient argument. Now look at $(2,0)$; There is a possible differential $H_2E \to H_2B \to H_1F$ by transgression. Then we have a short exact sequence $0 \to$ (some quotient of H_2E) $\to H_2(B) \to$ (a subgroup of H_2F) $\to 0$. Since the two side terms are in $\mathcal{C}$, so is the middle term. But we need some extra assumption to work on H_3B, since there is a nontrivial mapping from $(3,0) \to (1,1)$ on E^2. So unfortunately this strategy gets stuck here.

Now to make things work, let us add the extra assumption that $\mathcal{C}$ is either

1. closed under arbitrary sums, or
2. consisting entirely of finitely generated groups.

Consider case 1. if $\mathcal{C}$ has all infinite sums, then for $A \in \mathcal{C}$, B arbitrary, we would still have $A \otimes B, \mathrm{Tor}(A,B) \in \mathcal{C}$, so $H_i(X;A) \in \mathcal{C}$. So we have the following diagram mod $\mathcal{C}$:

$$\begin{array}{c|ccccc}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & ? & ? & ? & ? \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

If one is familiar with SSS, one can directly read off the SSS that $H_iE \to H_iB$ is iso mod $\mathcal{C}$ and thus prove the theorem. To be more detailed, let us look at the mapping $H_nB \xrightarrow{d_2} H_{n-2}(B;H_1F)$, and get the corresponding SES $0 \to \ker d_2 \to H_nB \to \operatorname{im} d_2 \to 0$. Since $\operatorname{im} d_2$ is a subgroup of something $0_{\mathcal{C}}$, so is itself $0_{\mathcal{C}}$; so it remains to prove that $\ker d_2$ is $0_{\mathcal{C}}$.

To do this, let us look at the E^∞ page. We see that H_nE, which is in $\mathcal{C}$, is the filtration of n groups (everything on the diagonal but the bottom one) that are in $\mathcal{C}$ already, along with the last one on the bottom row, which is then forced to be in $\mathcal{C}$ as well. Thus we can conclude that $E^\infty_{r,0} \in \mathcal{C}$ for arbitrary r. Suppose $E^\infty_{r,0} = E^m_{r,0} = \ker d_m / \operatorname{im} d_m$. Since $\operatorname{im} d_m \in \mathcal{C}$, we know that $\ker d_m \in \mathcal{C}$. Since $E^{m-1}_{r,0} / \ker d_m = \operatorname{im} d_m \in \mathcal{C}$, it follows that $E^{m-1}_{r,0} \in \mathcal{C}$, and continue this argument we eventually get that $\ker d_2 \in \mathcal{C}$ and thus $H_iB \in \mathcal{C}$.

We'll do case 2, and the applications that require it, in the next class. (We didn't; just assume it.) □

15 Mod C Hurewicz Theorem

Recall the definition of a Serre class. Clearly, if $E^2_{p,q} \in \mathcal{C}$, then so does $E^\infty_{p,q}$. Similarly, if everything along a diagonal n on the E^∞ page is in $\mathcal{C}$, then so is $H_n E$. (This is how Serre class interacts well with SSS.)

Theorem 15.1 (Mod $\mathcal{C}$ Hurewicz Theorem)**.** *Suppose $\mathcal{C}$ is a Serre class satisfying*

1. *$A, B \in \mathcal{C} \implies A \otimes B, Tor(A, B) \in \mathcal{C}$ (for instance, when $\mathcal{C}$ consists of fin. gen. abelian groups), and*
2. *$A \in \mathcal{C} \implies H_n(K(A, n); \mathbb{Z}) \in \mathcal{C}$ for each n.*

Suppose X is simply connected, then if $\pi_i = 0 \bmod \mathcal{C}$ for $i < n$, then $H_i X = 0 \bmod \mathcal{C}$ for $i < n$ and that $\pi_n X \to H_n X$ is an isomorphism mod $\mathcal{C}$.

Example 9. *If $\mathcal{C}$ is a collection of fin. gen. abelian groups staisfying the condition above, then if X is fin. gen. and $H_n X$ are fin. gen. for each n, then $\pi_n X$ are fin. gen.*

Corollary 6. *$\pi_n S^k$ is finitely generated for $k \geq 2$.*

In the next class, we'll prove the even cooler result that $\pi_n S^k$ is **finite** for $k > n$.

Proof. By induction on n. When $n = 2$ we have the usual Hurewicz theorem. So we may assume $n > 2$.

First Attempt Recall that in the regular proof, the inductive step was given by the following diagram:

$$\begin{tikzcd} \pi_i X \arrow[r] \arrow[d, "\cong"] & H_i X \arrow[d, "\cong"] \\ \pi_{i-1}\Omega X \arrow[r] & H_{i-1}\Omega X \end{tikzcd}$$

where the right equivalence holds in a range of i using SSS. We reduce dimension by 1 by going to the loopspace, then the inductive hypothesis settles the problem. However, note that given $\pi_i X = 0 \bmod \mathcal{C}$ for $i < n$ for some n does not implies $\pi_2 X = \pi_1 \Omega X = 0$, so we cannot assume ΩX is simply connected as we did in the regular proof, so we can't "bootstrap" this proof naively.

Hotfix To resolve this, consider the following fibration:

$$F \to X \to K(\pi_2 X, 2).$$

Then we have $\pi_{\leq 2} F = 0, \pi_i F = \pi_i X$ for $i > 2$. Then look at the following:

$$\Omega F \to \Omega X \to \Omega K(\pi_2 X, 2) = K(\pi_2 X, 1).$$

Then ΩF is simply connected, and $\pi_i \Omega F = \pi_i \Omega X$ for $i \geq 2$.
We will establish the following equivalences:

- $\pi_i X = \pi_{i-1}\Omega X = \pi_{i-1}\Omega F \bmod \mathcal{C}$ for all i, and
- $H_i X = H_{i-1}\Omega X = H_{i-1}\Omega F \bmod \mathcal{C}$ for $i \leq n$.

Then the inductive hypothesis applied to ΩF gives the result. The first equality is trivial by construction, so let's look at the second one.

Note By universal coefficient theorem, the two hypotheses on the Serre class together imply that for each n, $A, B \in \mathcal{C} \implies H_n(K(A, 1); B) \in \mathcal{C}$.

Second Equality Consider again the SES $\Omega F \to \Omega X \to K(\pi_2 X, 1)$. One can check that in a looped fibration, $\pi_1 B$ acts trivially on $H_i F$. So we can apply Serre spectral sequence without using local coefficients. Let us apply SSS.

$$
\begin{array}{c|ccccccc}
n & ? & ? & ? & ? & ? & ? & ? \\
n-1 & 0 & ? & ? & ? & ? & ? & ? \\
n-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

Consider the mod $\mathcal{C}$ SSS above. The 0th column now consists of homologies of ΩF, so until $H_{n-1}\Omega F$ these groups are in $\mathcal{C}$. We know that the 0th row is 0 mod $\mathcal{C}$, then from the universal coefficient theorem we see that the all the way to the row of $n-2$, everything below is 0 mod $\mathcal{C}$. Then since we know that the SSS converges to $H_n \Omega X$, by SSS we have that $H_i \Omega X = 0_{\mathcal{C}}$, i.e. $H_i \Omega F \to H_i \Omega X$ is iso mod $\mathcal{C}$ for $i \leq n-1$. This is the second half of the second point.

Now look at the fibration $\Omega X \to PX \to X$. By induction, $H_i X = 0$ mod $\mathcal{C}$ for $i < n-1$ because $\pi_i X = 0$ mod $\mathcal{C}$ for $i < n$. Also by induction, we know that $H_i \Omega X = H_i \Omega F = 0$ mod $\mathcal{C}$ for $i < n-1$. Now look at the SSS of the fibration above.

$$
\begin{array}{c|ccccccc}
n & ? & ? & ? & ? & ? & ? & ? \\
n-1 & ? & ? & ? & ? & ? & ? & ? \\
n-2 & 0 & 0 & 0 & 0 & 0 & ? & ? \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & ? & ? \\
\hline
 & 0 & 1 & 2 & \ldots & n-2 & n-1 & n
\end{array}
$$

By induction and universal coefficient, everything to the left of $(n-1)$ column and below $(n-1)$ row is 0 (except at $(0,0)$). Then looking at action of d_2 on $(n-1,0)$ (it injects into a $0_{\mathcal{C}}$) we see that $(n-1,0)$ is 0 mod $\mathcal{C}$, which clears the $(n-1)$ column, which then yields that $(n,0) \to (0,n-1)$ is iso mod $\mathcal{C}$. This yields the first half of the second point. $\square$

Proposition 11. *If A is finitely generated, then $H_n(K(A,1);\mathbb{Z})$ is finitely generated for all $n \geq 0$. If A is a p-torsion abelian group, then $H_n(K(A,1);\mathbb{Z})$ is p-torsion for all $n \geq 1$.*

Proof. First, for any abelian group A, A is a filtered colimit $A = \mathrm{colim}_{A' \subseteq A, A' \text{ finitely generated}} A'$. If A is p-torsion, then each A' is finitely generated and p-torsion. Since homology commutes with colimit, it suffices to prove the first part (and showing that the homologies are p-torsion). Now note that $K(A \times B, 1) = K(A,1) \times K(B,1)$. By Kunneth formula, we are reduced to $A = \mathbb{Z}$ and $A = \mathbb{Z}/p$. Now $K(\mathbb{Z},1) = S^1$, so no problem with $A = \mathbb{Z}$. By the theorem that $[K(A,m), K(B,m)] = \mathrm{Hom}(A,B)$ (proof can be found here), we know that $K(\mathbb{Z},2) \xrightarrow{p^n}$

$K(\mathbb{Z},2) \to K(\mathbb{Z}/p^n,2)$ is a fibration[5]. If we back it up twice, we have $S^1 \to K(\mathbb{Z}/p^n,1) \to K(\mathbb{Z},2) = \mathbb{CP}^\infty$. Now we look at what happens in cohomology SSS:

$$
\begin{tikzcd}[row sep=small, column sep=small]
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathbb{Z}_e \arrow[drr] & 0 & \mathbb{Z}_{ex} \arrow[drr] & 0 & \mathbb{Z}_{ex^2} \arrow[drr] & 0 & \mathbb{Z}_{ex^3} \\
0 & \mathbb{Z}_1 & 0 & \mathbb{Z}_x & 0 & \mathbb{Z}_{x^2} & 0 & \mathbb{Z}_{x^3} \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

where all the differentials are multiplications by p^n (induced by $K(\mathbb{Z},2) \xrightarrow{p^n} K(\mathbb{Z},2)$). Thus we have $H^*(K(\mathbb{Z}/p^n,1);\mathbb{Z}) = \mathbb{Z}[x]/(p^n x)$ for $|x| = 2$. By the universal coefficient theorem we have $H_{2m-1}(\mathbb{Z}/p^n,1);\mathbb{Z}) = \mathbb{Z}/p^n$ for $m \geq 1$. In particular, this means $H_0 = \mathbb{Z}, H_{2m} = 0$ for $m \geq 1$. $\square$

[5] I'm still not sure how to go about this, I think there might be some functor $K(-,n)$ or a delooping functor that preserves the fibration, but this is not very clear.

16 Relatvie Mod C Hurewicz; Finiteness of Homotopy Groups of Spheres

First, a warning: the assumption of simple connectivity is actually necessary. An example: take $S^2 \vee S^1$, and certainly the homology groups are finitely generated, and $\pi_1 = \mathbb{Z}$, but $\pi_2 = \mathbb{Z}^{\mathbb{Z}}$ by the universal cover of the graph (where S^1 expands to $\mathbb{Z}$, and $H_2 = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} = \pi_2$, which is not finitely generated) is not finitely generated

Again consider the backing-up of $S^3 \to K(\mathbb{Z}, 3)$, which is $K(\mathbb{Z}, 2) \to X \to S^3$. We used cohomology SSS on this to compute some homotopy groups of S^3. Now fix some prime p and let $\mathcal{C}$ be the class of ℓ torsion groups such that $(\ell, p) = 1$. Then we know that $H_{2p}(X) = \mathbb{Z}/p$ and $H_i X = 0 \bmod \mathcal{C}$ for $i < 2p$. Then by mod $\mathcal{C}$ Hurewicz, we have $\pi_i X = 0 \bmod \mathcal{C}$ for $i < 2p$ and $\pi_{2p}(X) = \mathbb{Z}/p \bmod \mathcal{C}$. Thus we know that the next mod $\mathcal{C}$ nontrivial fundamental group for S^3 after $\pi_3(S^3) = \mathbb{Z}$ is $\pi_{2p} S^3 = \mathbb{Z}/p \oplus A$ for A being a torsion group of order smaller than p (because larger factors aren't supposed to show up yet).

Corollary 7. *$\pi_i S^3 \neq 0$ for infinitely many i values (one for each p).*

Now let $\mathcal{C}$ be the Serre class of all torsion abelian groups. Then $H_i X = 0 \bmod \mathcal{C}$, so $\pi_i X = 0 \bmod \mathcal{C}$. Thus we know that $\pi_i S^3 = \mathbb{Z}$ for $i = 3$ and torsion when $i > 3$. But since π_i for $i > 3$ is also finitely generated (because the homology of S^3 is finitely generated, then we invoke mod $\mathcal{C}$ Hurewicz; note that this generalizes to the fact that any simply connected space with finitely generated homology (i.e. finitely many cells at each level) yields finitely generated homotopy), we conclude:

Corollary 8. *$\pi_i S^3$ is **finite** for $i > 3$.*

This idea of "looking at homotopy groups locally" turned out to be quite far-reaching and can be seen as a root for the development of chromatic homotopy theory.

16.1 Relative mod C Hurewicz Theorem

Theorem 16.1. *Let $A \hookrightarrow X$ be a pair of spaces, where X and A are both simply connected. Additionally, assume the assumption that we had in lecture 14. If $\pi_i(X, A) = 0 \bmod \mathcal{C}$ for $i < n$, then $H_i(X, A) = 0 \bmod \mathcal{C}$ for $i < n$, and $\pi_n(X, A) \to H_n(X, A)$ is iso mod $\mathcal{C}$.*

Proof. Let $F \to A \to X$ be the fibration. Invoke the Serre Spectral Sequence.

$$
\begin{array}{c|ccccccc}
n & ? & ? & ? & ? & ? & ? & ? \\
n-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & \dots & n-2 & n-1 & n
\end{array}
$$

Then we see that $H_i(A) \to H_i(X)$ is iso for $i < n-1$ and epi for $i = n-1$ mod$\mathcal{C}$, and thus $H_i(X, A) = 0 \bmod \mathcal{C}$ for $i < n$. The rest of the proof is similar as before and we skip. □

Suppose $\mathcal{C}$ is the class of torsion abelian groups.

Lemma 10. *A map $A \to B$ is iso mod $\mathcal{C}$ iff $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is iso.*

Proof. Easy. □

Observing this, it is easy to prove the following:

Proposition 12. *Suppose we have $f : X \to Y$ and that $H_*(X), H_*(Y)$ are finitely generated, X and Y are simply connected, then the following are equivalent:*

1. $H_*(X;\mathbb{Q}) \cong H_*(Y;\mathbb{Q})$;

2. $H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$;

3. $\pi_* X \to \pi_* Y$ *is iso mod* $\mathcal{C}$; *and*

4. $\pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q}$ *is iso mod* $\mathcal{C}$.

Now recall that $H^*(K(\mathbb{Z},1);G) = H^*(S^1;G) = G[x]/(x^2)$ for $|x| = 1$. Now look at the CSSS of $K(\mathbb{Z},1)$ and $K(\mathbb{Z},2)$ (i.e. $K(\mathbb{Z},1) \to PK(\mathbb{Z},2) \to K(\mathbb{Z},2)$ with rational coefficients: cohomology of $K(\mathbb{Z},2)$ $(\mathbb{Q},0,\mathbb{Q},0,\dots)$ is the 0th row, that of $K(\mathbb{Z},1)$ $(\mathbb{Q},\mathbb{Q},0,0,\dots)$ is 0th column. Then by staring at the CSSS we get that $H^*(K(2,\mathbb{Z});\mathbb{Q}) = \mathbb{Q}_{x_2}$ for some $|x_2| = 2$. Do this again, we see $K(\mathbb{Z},3)$ has $\mathbb{Q}$-cohomology ring $\mathbb{Q}[x_3]/x_1^2$. Keep repeating this, we conclude that $H^*(K(\mathbb{Z},2n);\mathbb{Q})$ is $\mathbb{Q}[x_{2n}]$, and $H^*(K(\mathbb{Z},2n+1);\mathbb{Q}) = \mathbb{Q}[x_{2n+1}]/x_{2n+1}^2$ is an exterior algebra.

Note that $S^{2n-1} \to K(\mathbb{Z},2n-1)$ is an iso in $H^*(\bullet;\mathbb{Q})$, and thus an iso in $\pi_* \bullet \pmod{\mathcal{C}}$ as the proposition above indicates, and thus $\pi_i(S^{2n-1})$ is finite for $i > 2n-1$. ($\pi_i(S^{2n-1})$ is trivial for $i < 2n-1$.)

What about the other half of the possibility, i.e. S^n for n even? We have $S^{2n} \to K(\mathbb{Z},2n)$, which is $\mathbb{Q}[e]/e^2 \to \mathbb{Q}[x_{2n}]$ explicitly given by $e \mapsto x_{2n}$ where $|e| = 2n$. Now we have a fibration $S^{2n} \to K(\mathbb{Z},2n) \xrightarrow{x_{2n}^2} K(\mathbb{Z},4n)$. Look at the Serre spectral sequence of the backup $K(\mathbb{Z},4n-1) \to S^{2n} \to K(\mathbb{Z},2n)$ to get that $H^*(S^{2n};\mathbb{Q}) = \mathbb{Q}[x_{2n},x_{4n-1}]/(x_{2n}^2,x_{4n-1}^2)$. We conclude that $\pi_* S^{2n} \otimes \mathbb{Q}$ is $\mathbb{Q}$ for $* = 2n$, $\mathbb{Q}$ for $* = 4n-1$, and 0 otherwise. As a corollary, $\pi_{2n} S^{2n} = \mathbb{Z}$, $\pi_{4n-1} S^{2n} = \mathbb{Z} \oplus$ a finite group, and $\pi_i S^{2n}$ is finite for other is.

Note that there is a systematic treatment of rational homotopy theory via model category theory, due to Quillen.

17 Introduction to Steenrod Operations

Problem 3 (Steenrod's Problem)**.** *If X is a CW complex of dimension $\leq n+1$, then what is $[X, S^n]$?*

Definition 20. *$W \to Z$ is an n-equivalence between 1-connected spaces if*

1. *$\pi_i(Z, W) = 0$ for $i \leq n$,*
2. *$H_i(Z, W) = 0$ for $i \leq n$,*
3. *$\pi_i W \to \pi_i Z$ is isomorphism for $i < n$ and epimorphism for $i = n$, and*
4. *$H_i W \to H_i Z$ is the same.*

Proposition 13. *If $W \to Z$ is an n-equivalence and X is a CW-complex, then $[X, W] \to [X, Z]$ is an isomorphism for $\dim X < n$ and epimorphism for $\dim X = n$.*

Proof. First let's show the epimorphism. Consider the following diagram, where X is assumed to have dimension n, and let $X^{(n-1)}$ be its $(n-1)$-skeleton.

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[r] \arrow[d] & W \arrow[d] \\
D^n \arrow[r] \arrow[urr, dotted] & X \arrow[r] & Z
\end{tikzcd}$$

Since $W \to Z$ is $(n-1)$-equivalent, it follows induction that every map in $[X^{(n-1)}, Z]$ comes from some map in $[X^{(n-1)}, W]$, so it suffices to deform all the mapping $D^n \to X \to Z$ to $D^n \to W$. Consider the following diagram where W' is the pullback of the two maps on the right:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & W \arrow[d] \\
D^n \arrow[r] \arrow[ur, dotted] & D^n \arrow[r] & Z
\end{tikzcd}$$

Then as one can check, W' and D^n has n-equivalence, and thus isomorphic for π_{n-1}, thus $S^{n-1} \to W'$ is nulhomotopic, so it extends to some $D^n \to W'$. Then the concatenation with $W' \to W$ gives us the required lifting (at least up to homotopy), and thus we can conclude that every $X \to Z$ can be deformed to some $X \to W$ for $\dim X = n$.

Now we look at the isomorphism claim. Consider the following graph:

$$\begin{tikzcd}
X \times I \arrow[r] \arrow[d] & W \arrow[d] \\
X \times I \arrow[r] \arrow[ur, dotted] & Z
\end{tikzcd}$$

Suppose $\dim(X) < n$, then all cells of $X \times I$ have dimension $\leq n$, so as we have argued above, the lifting $X \times I \to W$ exists, i.e. any homotopy in Z lifts to a homotopy in W. Thus maps that are zero in $[X, Z]$ will be zero in the lifting, hence the monomorphism and isomorphism. □

Now let us address Steenrod's problem. Suppose we have $S^n \to K(\mathbb{Z}, n)$, then clearly it's an iso on π_i for $i \leq n$, and epi for $i = n+1$. This tells us that if $\dim X < n+1$, then $[X, S^n] = [X, K(\mathbb{Z}, n)] = H^n(X; \mathbb{Z})$. In short, two maps of the same degree are homotopic (consider e.g. oriented n-manifolds). This result (or at least its earlier versions) was known in the 1930s. Also, this was part of the reason why cohomology was introduced.

On the other hand, we also know that the mapping $S^n \to K(\mathbb{Z}, n)$ is iso on $H_i, i \leq n$ and epi on H_{n+1}. Thus we know that $H_n(K(\mathbb{Z}, n); \mathbb{Z}) = \mathbb{Z}$ and $H_{n+1}(K(\mathbb{Z}, n); \mathbb{Z}) = 0$. But if we want to go further, we need to figure out H_{n+2}. This motivates our further study on the (co)homology of Eilenberg-Maclane spaces.

Let's try to figure out the cohomology of $K(\mathbb{Z}, 3)$ with SSS on its path fibration:

$$K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3).$$

$$
\begin{array}{c|ccccccc}
4 & \mathbb{Z}_{x^2} & 0 & 0 & \mathbb{Z}_{x^2 i} & 0 & 0 & ? \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & ? \\
2 & \mathbb{Z}_{x} & 0 & 0 & \mathbb{Z}_{xi} & 0 & 0 & ? \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & ? \\
0 & \mathbb{Z}_{1} & 0 & 0 & \mathbb{Z}_{i} & 0 & 0 & \alpha \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

In the CSSS above, α is some element such that $\alpha^2 = 0$ and is killed by xi. Then we can conclude that $H^3(K(\mathbb{Z}, 3)) = \mathbb{Z}$, $H^6(K(\mathbb{Z}, 3)) = \mathbb{Z}/2$, and further groups are 0. Using UCT, we have that $H_3(K(\mathbb{Z}, 3)) = \mathbb{Z}, H_5 = \mathbb{Z}_2$. (See here for the details of this computation.) Or we can look at $\pi_i(K(\mathbb{Z}, 3), S^3) = 0$ for $i < 5$ and $\mathbb{Z}/2$ when $i = 5$ (which follows the homotopy LES). Then this yields that $H_5(K(\mathbb{Z}, 3)) = \mathbb{Z}_2$, from which the same result can be deduced.

Now consider the SSS for the next fibration in line:

$$K(\mathbb{Z}, 3) \to * \to K(\mathbb{Z}, 4).$$

$$
\begin{array}{c|ccccccc}
5 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z}_2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

where $(0, 6)$ comes from transgression. Continuing this process, we conclude that $H_{n+2}(K(\mathbb{Z}, n); \mathbb{Z}) = \mathbb{Z}_2$.

17.1 Motivating Steenrod Squares

Now we take a more geometric point of view and start considering the chain complex $C_*K(\mathbb{Z}, n)$ associated with the Eilenberg-Maclane space. By cellular homology, we can guess the following structure:

$$\ldots \to C_{n+3} = \mathbb{Z} \xrightarrow{2} C_{n+2} = \mathbb{Z} \to C_{n+1} = 0 \to C_n = \mathbb{Z} \to \ldots$$

We want to map $K(\mathbb{Z}, n)$ into something else so that we can get rid of the $\mathbb{Z}_2$ homology at $n + 2$.

Naive Approach For the rest of this discussion, keep in mind of the general fact that $[W, K(A, m)] = H^m(W; A)$. Consider $S^n \to K(\mathbb{Z}, n)$ and extend it to $S^n \to K(\mathbb{Z}, n) \to K(\mathbb{Z}, n + 3)$ and take the homotopy fiber $F \to K(\mathbb{Z}, n) \to K(\mathbb{Z}, n + 3)$, so that $S^n \to K(\mathbb{Z}, n)$ factors through F as follows:

$$
\begin{tikzcd}
F \arrow[r] & K(\mathbb{Z}, n) \arrow[r] & K(\mathbb{Z}, n+3) \\
S^n \arrow[u] \arrow[ur] & &
\end{tikzcd}
$$

Take it back a step to $K(\mathbb{Z}, n+2) \to F \to K(\mathbb{Z}, n)$, and consider the SSS. If we look at the SSS, we see that $\mathbb{Z}$ at $(0, n+2)$ maps to $\mathbb{Z}_2$ at $(n+3, 0)$, so it actually does not kill the $\mathbb{Z}_2$. We want a $\mathbb{Z}_2$ at $(0, n+2)$.

However, naively using $K(\mathbb{Z}_2, *)$ doesn't work either because $H^{n+2}(K(\mathbb{Z}_2, n+2); \mathbb{Z}) = 0$ and $H^{n+3}(K(\mathbb{Z}_2, n+2); \mathbb{Z}) = \mathbb{Z}_2$.

Solution Recall that $H^n(K(\mathbb{Z}, n); \mathbb{Z}) = \mathbb{Z}$ and $H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}_2) = \mathbb{Z}_2$. So instead we consider the fibration $K(\mathbb{Z}_2, n+1) \to F \to K(\mathbb{Z}, n)$ obtained from backing up the fibration $F \to K(\mathbb{Z}, n) \to K(\mathbb{Z}_2, n+2)$.

Before we continue, a few words about $[X, F]$. First, observe that for $\dim X = n+1$ we have $[X, S^n] \to [X, F]$ being an isomorphism. Additionally, $S^n \vee S^n$ and $S^n \times S^n$ have cohomology agree up to $2n$, but the first one has a natural map onto S^n, so $[X, S^n]$ has a group structure for $\dim X$ up to $2n$. If we localize mod p, on the other hand, then we actually have a group structure all the way up.

Now consider the long exact sequence $[X, K(\mathbb{Z}, n-1)] \to [X, K(\mathbb{Z}/2, n+1)] \to [X, F] \to [X, K(\mathbb{Z}, n)] \to [X, K(\mathbb{Z}/2, n+2)] \to \ldots$. This yields $H^{n-1}(X; \mathbb{Z}) \to H^{n+1}(X; \mathbb{Z}/2) \to [X, F] \to H^n(X; \mathbb{Z}) \to 0$. Note that we "magically" now have a mapping (the first one) that increases the cohomology degree by 2 and reduces the coefficient mod 2. This is a **new structure** in the cohomology, denoted by $Sq^2 : H^k(X; \mathbb{Z}) \to H^{k+2}(X; \mathbb{Z}/2)$.

Proposition 14. *The Steenrod square Sq^2 has the following properties:*

1. *additivity,*
2. $x \in H^2(X) \implies Sq^2(x) = x^2 \pmod 2$, *and*
3. $Sq^2(xy) = Sq^2(x)y + xSq^2(y)$.

Example 10. *Consider $[\mathbb{CP}^m, S^{2m-1}]$. We have the long exact sequence $H^{2m-2}(\mathbb{CP}^m; \mathbb{Z}) \to H^{2m}(\mathbb{CP}^m; \mathbb{Z}_2) \to [\mathbb{CP}^m, S^{2m-1}] \to H^{2m-1}(\mathbb{CP}^m; \mathbb{Z}) = 0$. The first one is $\mathbb{Z}$ generated by x^{m-1} (where x is a generator of $H^2(\mathbb{CP}^m; \mathbb{Z})$), the second one is $\mathbb{Z}_2$; by the Leibniz's rule above, we see that the map sends x^{m-1} to $Sq^2(x^{m-1}) = (m-1)x^m$, so the mapping is multiplication by $(m-1)$. Thus we conclude that $[\mathbb{CP}^m, S^{2m-1}] = \mathbb{Z}_2$ for m odd, and 0 for m even.*

Eventually we shall be able to calculate the cohomology of Eilenberg-Maclane spaces using this machinery (this was done by Cartan and Serre).

18 Construction of Steenrod Operations

Let's start with Brown's representability theorem. We know that there is an natural isomorphism $[X, K(A,n)] \equiv H^n(X, A)$, so we shall not differentiate between the two structures and abuse the notation a bit in what follows.

Recall the external cup product used in Kunneth formula: $H^i(X) \times H^j(Y) \to H^i(X) \otimes H^j(Y) \to H^{i+j}(X \times Y)$. Given two maps $f : [X, K(A,n)]$ and $g : [X, K(A,m)]$, we shall write $f \cup g$ to represent their image under this external cup product. In particular, let a, b be two mappings $X \to K(\mathbb{Z}_2, n)$ and $X \to K(\mathbb{Z}_2, m)$ respectively, and let i_m be the canonical mapping $K(A,n) \to K(A, n+m)$. The the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "\Delta"] \arrow[ddr, "a \cup b"] & \\
X \times X \arrow[d, "a \times b"] \arrow[dr] & \\
K(\mathbb{Z}_2, n) \times K(\mathbb{Z}_2, m) \arrow[r, "i_m \cup i_n"] & K(\mathbb{Z}_2, n+m)
\end{tikzcd}$$

Note that $[a \cup b] = [b \cup a]$ as cohomology classes, which is to say that (by abuse of notation) as maps, $a \cup b$ and $b \cup a$ are homotopic. The important observation that Steenrod made was that there is much information in this homotopy.

Consider the following diagram, given $a : X \to K(\mathbb{Z}_2, n)$. This diagram is commutative up to homotopy:

$$\begin{tikzcd}
X \times X \arrow[r, "a \cup a"] \arrow[d, "(x,y) \mapsto (y,x)"'] & K(\mathbb{Z}_2, 2n) \\
X \times X \arrow[ur, "a \cup a"] &
\end{tikzcd}$$

This means that we have a homotopy expressible as $H : X \times X \times I \to K(\mathbb{Z}_2, 2n)$ such that $H(x, x', 0) = H(x', x, 1)$, therefore in particular H factors through space $X \times X \times I / \{(x, y, 0) \sim (y, x, 1)\}$. With a bit of imagination, observe that this space is homeomorphic to $S^1 \times X \times X / \mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ sends (λ, x, y) to $(-\lambda, y, x)$. We write this space as $S^1 \times_{\mathbb{Z}_2} X \times X$, so we have the mapping $h : S^1 \times_{\mathbb{Z}_2} X \times X \to K(\mathbb{Z}_2, 2n)$.

In general, let W be a space with an action of $\mathbb{Z}_2$, and let $\tau \neq 0 \in \mathbb{Z}_2$. Then a mapping $S^1 \times_{\mathbb{Z}_2} W \xrightarrow{h} Z$ is the same thing as a map $W \xrightarrow{g} Z$ along with a homotopy $g \cong g \circ \tau$.

Later we will prove that the map $h : S^1 \times_{\mathbb{Z}_2} X \to X \to K(\mathbb{Z}_2, 2n)$ extends to $D^2 \times X \times X \to K(\mathbb{Z}_2, 2n)$, which then extends to $S^2 \times_{\mathbb{Z}_2} X \times X \to K(\mathbb{Z}_2, 2n)$, etc. Eventually it extends to $S^\infty \times_{\mathbb{Z}_2} X \times X \to K(\mathbb{Z}_2, 2n)$. This is the main idea: we look for a map $S^\infty \times_{\mathbb{Z}_2} X \times X \xrightarrow{\tilde{p}(a)} K(\mathbb{Z}_2, 2n)$ extending $X \times X \xrightarrow{a \cup a} K(\mathbb{Z}_2, 2n)$. In particular, note that $\tilde{p}(a) \in H^{2n}(S^\infty \times_{\mathbb{Z}_2} X \times X; \mathbb{Z}_2)$.

Theorem 18.1. *For any pair (X, A), there is a unique natural transformation $H^n(X, A; \mathbb{Z}_2) \xrightarrow{\tilde{p}} H^{2n}(S^\infty \times_{\mathbb{Z}_2} (X, A)^2; \mathbb{Z}_2)$, such that the following diagram commutes*[6]*:*

$$\begin{tikzcd}
H^n(X, A; \mathbb{Z}_2) \arrow[r, "\tilde{p}"] \arrow[dr, "a \mapsto a \cup a"'] & H^{2n}(S^\infty \times_{\mathbb{Z}_2} (X, A)^2; \mathbb{Z}_2) \arrow[d] \\
& H^{2n}((X, A)^2; \mathbb{Z}_2)
\end{tikzcd}$$

where $(X, A)^2 = (X \times X, X \times A \cup_{A \times A} A \times X)$.

Constructing Steenrod Operations First, from now on, as long as we're talking about Steenrod algebra, one should always assume the coefficient group to be $\mathbb{Z}_2$. Start with the diagonal cup product $H^n(X) \xrightarrow{\Delta} H^n(X) \otimes H^n(X) \to H^{2n}(X \times X)$. Observe that we have the canonical map $h : \mathbb{RP}^\infty \times X \to S^\infty \times_{\mathbb{Z}_2} X \xrightarrow{\Delta} S^\infty \times_{\mathbb{Z}_2} X \times X$ (note that this is well-defined since the image of the diagonal map is invariant under the $\mathbb{Z}_2$ action). Assuming the theorem above, we get the following factorization of the diagonal cup product:

$$\begin{tikzcd}
H^n(X) \arrow[r, "\tilde{p}"] \arrow[dr, "P"] & H^{2n}(S^\infty \times_{\mathbb{Z}_2} X \times X) \arrow[r] \arrow[d, "h^*"] & H^{2n}(X \times X) \arrow[d, "\Delta^*"] \\
& H^{2n}(\mathbb{RP}^\infty \times X) \arrow[r] & H^{2n}(X)
\end{tikzcd}$$

[6] For the vertical map, see next lecture.

The diagonal map P is often called the **total power operation**. Observe that its codomain is $H^*(\mathbb{RP}^\infty \times X) = \mathbb{Z}_2[t] \otimes H^*(X) = H^*(X)[t]$ for $|t| = 1$. It's important because it yields a **construction** of the Steenrod operations:

$$P(x) = x^2 + Sq^{n-1}(x)t + \ldots + Sq^0(x)t^n,$$

where we define $Sq^j (0 \le j \le n)$ as the coefficient of t^{n-j} in the expression above. In particular, $Sq^n(x) = x^2$.

Proof of Theorem 18.1. Now we describe this $\tilde{p}$. Given any (X, A) and a class $a \in H^n(X, A)$, there is a map $(X, A) \to K(\mathbb{Z}_2, n), *)$ such that a is pulled back from the canonical mapping $i_n \in H^n(K(\mathbb{Z}_2, n), *)$, therefore it suffices to show that there is a unique choice $\tilde{p}(i^n) \in H^{2n}(S^\infty \times_{\mathbb{Z}_2} (K(\mathbb{Z}_2, n), *)^2)$, and the rest of the proof follows from naturality. By Kunneth formula, we know that $H^{2n}((K(\mathbb{Z}_2, n), *)^2) = H^n(K(\mathbb{Z}_2, n))^{\otimes 2} = \mathbb{Z}_2$, so it suffices to show that the mapping $H^{2n}(S^\infty \times_{\mathbb{Z}_2} (K(\mathbb{Z}_2, n), *)^2) \to H^{2n}((K(\mathbb{Z}_2, n), *)^2)$ is an isomorphism. This follows from the following proposition immediately. □

Proposition 15. *Suppose* (X, A) *is* $(n-1)$*-connected, then there is a SES:*

$$0 \to H^{2n}(S^\infty \times_{\mathbb{Z}_2} (X, A)^2) \to H^{2m}((X, A)^2) = H^n(X, A) \otimes H^n(X, A) \xrightarrow{1-flip} H^n(X, A) \otimes H^n(X, A) \to 0.$$

Proof. We first show that for every $k \ge 1$, the map $S^k \times_{\mathbb{Z}_2} (X, A)^2 \to S^{k+1} \times_{\mathbb{Z}_2} (X, A)^2$ induces an isomorphism on $H^{2n}(\bullet; \mathbb{Z}_2)$. In particular, we need $H^{2n}(S^{k+1, S_k} \times_{\mathbb{Z}_2} (X, A)^2; \mathbb{Z}_2) = H^{2n+1}(S^{k+1, S_k} \times_{\mathbb{Z}_2} (X, A)^2; \mathbb{Z}_2) = 0$. The pair in consideration is relative homeomorphic to $(D^{k+1} \times \mathbb{Z}_2, S^k \times \mathbb{Z}_2) \times_{\mathbb{Z}_2} (X, A)^2$[7]; since both pairs are excisive, this induces isomorphism on cohomology. But note that $(D^{k+1} \times \mathbb{Z}_2, S^k \times \mathbb{Z}_2) \times_{\mathbb{Z}_2} (X, A)^2 \cong (D^{k+1}, S^k) \times (X, A)^2$, and the cohomology for the latter is $H^*(D^{k+1}, S^k) \otimes H^*(X, A)^{\otimes 2}$, which is 0 for degree less than $2n+2$ (by relative Hurewicz theorem).

Now that we have confirmed that the degree of the sphere doesn't matter, we verify the original claim with S^∞ replaced with S^1. Apply the same argument as above, but this time with the pair (S^1, S^0). We have the following maps:

$$\ldots \leftarrow S^0 \times_{\mathbb{Z}_2} (X, A)^2 \leftarrow S^1 \times_{\mathbb{Z}_2} (X, A)^2 \leftarrow (S^1, S^0) \times_{\mathbb{Z}_2} (X, A)^2 \leftarrow \ldots$$

where, by the same argument above, $H^{2n}((S^1, S^0) \times_{\mathbb{Z}_2} (X, A)^2) = H^{2n}(S^1 \times (X, A)^2) = 0$, so we have the following map

$$0 \to H^{2n}(S^1 \times_{\mathbb{Z}_2} (X, A)^2) \to H^{2n}(S^0 \times_{\mathbb{Z}_2} (X, A)^2) \to \ldots,$$

so it remains to check that the next map on the first sequence is $1 - \text{flip}$, which is a boilerplate definition check. □

[7] details of this homeomorphism can be found here.

19 More on Steenrod Operations

Last time we showed that if (X, A) is $(n-1)$-connected, then $H^i(S^\infty \times_{\mathbb{Z}_2} (X, A)^2) = 0$ for $i < 2n$, and $H^{2n}(S^\infty \times_{\mathbb{Z}_2} (X, A)^2)$ are the elements in $H^n(X, A)^{\otimes 2}$ invariant under the tensor factor transposition.

In particular, let $(X, A) = K_n$, then we have a corollary that there is a unique element $\beta(i_n) \in H^{2n}(S^\infty \times_{\mathbb{Z}_2} K_n^2)$ restricting to $i_n \otimes i_n$. Thus, the P operator that we mentioned last time uniquely defines $Sq^i : H^n(X) \to H^{n+i}(X)$ for $i \leq n$.

19.1 Properties of the Steenrod Operations

Steenrod operations have many nice properties (all products are cup products, and $x \in H^n(X)$):

1. Sq^i are homomorphisms.
2. $Sq^n(x) = x^2$.
3. $Sq^0(x) = x$.[8]
4. Cartan formula: $Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y)$.
5. $Sq^i(x) = 0$ for all $i < 0$.
6. Stability, i.e. the following square commutes:

$$\begin{tikzcd} H^m(X) \arrow[r] \arrow[d] & Sq^n H^{m+n}(X) \arrow[d] \\ H^m(\Sigma X) \arrow[r] & Sq^n H^{m+n}(\Sigma X) \end{tikzcd}$$

7. $Sq^n(x) = 0$ for $n > \dim(X)$.

In addition, all of these properties are natural. On Friday, these properties uniquely identify the Steenrod algebra (Turned out we didn't.)

Proof. Now let's prove each of these properties.

Claim 1 and 2 We have shown those above in the construction.

Claim 5 Suppose $x \in H^n(X; \mathbb{Z}_2)$ and $i < 0$, then $Sq^i(x) \in H^{n+i}(x)$ for $n + i < n$. By naturality, this should hold for the case of $K(\mathbb{Z}_2, n)$ as well. However, consider the following square with respect to $\iota_n : X \to K(\mathbb{Z}_2, n)$:

$$\begin{tikzcd} x \in H^n(X) \arrow[d, "Sq^i"] & H^n(K(\mathbb{Z}_2, n)) \arrow[l, "\iota_n"'] \arrow[d, "Sq^i"] \\ H^{n+i}(X) & H^{n+i}(K(\mathbb{Z}_2, n)) \arrow[l] \end{tikzcd}$$

But when $n + i < n$, the right bottom object is zero, as Eilenberg-Maclane spaces have vanishing lower cohomologies, and thus we must have $Sq^i(x) = 0$.

Claim 4 Equivalently, it suffices to show $P(xy) = P(x)P(y)$, as the Cartan formula then follows from comparing coefficients. Note that x pulls back from some $i_n \in H^n K_n$, y pulls back from some $i_m \in H^m K_m$. Now consider the following diagram in the universal case (which we don't know commutes yet), in which we use the shorthand $K_j = (K(\mathbb{Z}_2, j), *)$:

[8] This one requires explicit computation on cohomology, and is not derivable from the algebraic structure of Steenrod algebra alone. For instance, in the algebraic construction of Steenrod algebra over a Galois field, the Sq^0 in fact corresponds to the Frobenius map.

$$\begin{tikzcd}
H^n(K_n) \times H^m(K_m) \arrow[d, "\cup"] \arrow[drr, "\tilde{p}\times\tilde{p}"] & & \\
H^{n+m}(K_n \times K_m) \arrow[d, "\tilde{p}"] & & H^{2n}(S^\infty \times_{\mathbb{Z}_2} K_n^2) \times H^{2m}(S^\infty \times_{\mathbb{Z}_2} K_m^2) \arrow[dll, "\cup"] \arrow[d, "h^* \times h^*"] \\
H^{2n+2m}(S^\infty \times_{\mathbb{Z}_2} (K_n \times K_m)^2) = \mathbb{Z}_2 \arrow[d, "h^*"] & & H^{2n}(K_n^2) \times H^{2m}(K_m^2) \arrow[dll, "\cup"] \\
H^{2n+2m}((K_n \times K_m)^2) & &
\end{tikzcd}$$

As a general observation, if (X, A) is $(n-1)$-connected and $H^k(X, A) = \mathbb{Z}_2$, then $H^{2k}(S^\infty \times_{\mathbb{Z}_2} (X, A)^2) = \mathbb{Z}_2$ because it is a subgroup of $H^k(X, A)^{\otimes 2} = \mathbb{Z}_2 \times \mathbb{Z}_2$ invariant under the flip isomorphism.

Thus we see the object second-to-bottom on the left column above is nothing other than $\mathbb{Z}_2$. On the other hand, since the lower rectangle commutes, we see that both maps factor through $\mathbb{Z}_2$ followed by h^*, so the image in the last row is $\mathbb{Z}_2$ (it's not 0 otherwise $P = 0$, which we refute later). Now in the universal case, the mapping $(i_m, i_n) \mapsto P(i_m \cup i_n)$ and $(i_m, i_n) \mapsto P(i_m) \cup P(i_n)$ both are mappings in $\mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, so either they are equal (and both nonzero), or at least one of them is the zero map. Since the general case pulls back from the universal case, we conclude that either $P(xy) = P(x)P(y)$ for all x, y, or one of them is constantly zero. So we can prove the theorem as long as we can see that both $(x, y) \mapsto P(xy)$ and $(x, y) \mapsto P(x)P(y)$ are nonzero at some point.

To do this, consider $H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[t]$. We know that $Sq^{n+m}(t^n t^m) = Sq^{n+m}(t^{n+m}) = t^{2(n+m)}$. On the other hand, $Sq^n(t^n) = Sq^m(t^m) = t^{2(n+m)}$, so neither is zero, and we have thus obtained the Cartan formula.

Claim 3 Again let's work with the universal case. Observe that $Sq^0(i_n) \in H^n(K(\mathbb{Z}_2, n), \mathbb{Z}_2) = \mathbb{Z}_2$. Moreover, any homomorphism $Sq^0 : H^n(X) \to H^n(X)$ is given, in the universal case, by a map in $[K(\mathbb{Z}_2, n), K(\mathbb{Z}_2, n)] \cong H^n(K(\mathbb{Z}_2, n), \mathbb{Z}_2) = \mathbb{Z}_2$, so there are only two natural maps: the identity map, and the zero map. Thus it suffices to show that Sq^0 is nonzero somewhere.

The example that we consider is $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. Again look at $h : S^\infty \times_{\mathbb{Z}_2} (\mathbb{R}^n, \mathbb{R}^n - \{0\})^2 \leftarrow \mathbb{RP}^\infty \times (\mathbb{R}^n, \mathbb{R}^n - \{0\}))$. So let's figure out what happens to H^{2n} of the first space. Note that H^{2n} of the second space is $H^n(\mathbb{RP}^\infty) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \mathbb{Z}_2$, which is generated by $t^n x$ for some nonzero generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. We would like to show that $\tilde{p}(x)$ gets map to $t^n x$.

In general, note that $(V, V - \{0\}) \times (W, W - \{0\}) = (V \times W, V \times W - \{0\})$. In particular, $(R^n, R^n - \{0\})^2 = (R^{2n}, R^{2n} - \{0\})$. Now consider how $\mathbb{R}^n$s embed into $\mathbb{R}^{2n}$: they embed as the "axes", and the $\mathbb{Z}_2$ factor will act by flipping these two axes (i.e. flipping around the "diagonal"). However, we can choose an alternative basis by choosing the diagonals as the "axes" (i.e. a different embedding), then we see that $\mathbb{Z}_2$ acts as -1 on one of them, and 1 on the other. This allows us to rewirte the mapping as $(S^\infty \times_{\mathbb{Z}_2} (\mathbb{R}^n_+, \mathbb{R}^n_+ - \{0\})) \times (\mathbb{R}^n_-, \mathbb{R}^n_- - \{0\}) \leftarrow \mathbb{RP}^\infty \times (\mathbb{R}^n, \mathbb{R}^n - \{0\}))$, in which case $\mathbb{Z}_2$ now does not act on the first parenthesis, so we have factored this map into product of two maps, where the second factor is the identity, and the firt factor is $RP^\infty = S^\infty \times_{\mathbb{Z}_2} (\{0\}, \emptyset) \to S^\infty \times_{\mathbb{Z}_2} (\mathbb{R}^n, \mathbb{R}^n - \{0\})$. By Kunneth formula, we are reduced to showing that this factor map is nonzero in H^n.

Before we proceed further, however, observe that $Sq^0(xy) = Sq^0(x)Sq^0(y)$, so it suffices to assume that $n = 1$. Now note $S^\infty \times_{\mathbb{Z}_2} (\mathbb{R}^1, \mathbb{R}^1 - 0)$ has the same cohomology as $S^\infty \times_{\mathbb{Z}_2} ((-1, 1), -1, 1)$ by a direct transformation. We want to show that $H^*(S^\infty \times_{\mathbb{Z}_2} ((-1, 1), -1, 1)) \to H^*(S^\infty \times_{\mathbb{Z}_2} (-1, 1))$ is an isomorphism. By the long exact sequence of a pair, it suffices to note that $H^1(S^\infty \times_{\mathbb{Z}_2} -1, 1) = H^1(S^\infty) = 0$.

Claim 6 and 7 are formal consequences of the ones above.

□

20 Mod 2 Cohomology of K(Z2, n)

We still want to compute the cohomology of Eilenberg-Maclance spaces. As we said before, the $\mathbb{Z}$-coefficient case is rather tricky, but today we can establish the $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ case. (From now on, omit the $\mathbb{Z}_2$ coefficient.)

Let's start with the only one that we're sure of. $H^*(K(\mathbb{Z}_2, 1); \mathbb{Z}_2) = H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{F}_2[x]$ for some $x \in H^1$. Now we want to consider $H^*(K(\mathbb{Z}_2, 2))$. Consider the fibration $K(\mathbb{Z}_2, 1) \to * \to K(\mathbb{Z}_2, 2)$ and its SSS (we'll only write generators now).

$$
\begin{array}{c|ccccccc}
5 & x^5 & 0 & ? & ? & ? & ? & ? \\
4 & x^4 & 0 & ? & ? & ? & ? & ? \\
3 & x^3 & 0 & ? & ? & ? & ? & ? \\
2 & x^2 & 0 & x^2y_2 & ? & ? & ? & ? \\
1 & x & 0 & xy^2 & ? & ? & ? & ? \\
0 & 1 & 0 & y_2 & y_3 & y_2^2 & ? & ? \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

The first column is given above. We know that $(1, 0)$ is zero and in fact 1st column is zero since $K(\mathbb{Z}_2, 2)$ is simply connected. By transgression, we must have $d_2(x) = y_2$ for some y_2. We know that $d_2(x^2) = 2xd_2(x) = 0$, so x^2 does not hit $(2, 1)$ and thus the mapping $d_2(xy_2) = y_2^2$ must be nonzero. Now, we know that $d_2(x^3)$ targets $x^2d_2(x) = x^2y_2$. On the other hand, x^2 must be transgressive, so $d_3(x^2) = y_3$ for some $y_3 \in (3, 0)$, which might be zero or might not. Technically, you can do this to figure out the rest of the E^2 page and thus the entire spectral sequence (see your homework).

The important thing to figure out is the differentials on x^{2^k}. We (i.e. the students) shall show the following:

Proposition 16. *Let $F \to E \to B$ be a fibration and let $\pi_1 B$ acts trivially on the fiber, and $H^*(E) = H^*(pt)$.*

1. *If $x \in H^*(F)$ is transgressive, then so is x^{2^n} for all n.*
2. ***Borel's Theorem** (see homework) Assume cohomology works on $\mathbb{Z}_2$ coefficients. The cohomology of B is given by the polynomial algebra generated by the images of all the transgressions.*

We shall let the target of x^{2^n} be called y^{2^n+1}. The proof of Borel's theorem is done in the homework using Zeeman comparison theorem,

Koszul Resolution Consider the abstract chain complex $C_* = \mathbb{F}_2[x][y_2, y_3, \ldots, y_{2^n+1}, \ldots]$. Define the differential $d(x^{2^n}) = y_{2^n+1}$. Then this extends to all x^n in the obvious manner: e.g. $d(x^{14}) = d^(x^8 \cdot x^4 \cdot x^2) =$ (Leibniz Rule) $= x^12y_3 + x^{10}y_5 + x^6y_9$. More abstractly, we want a map from vector space $\bigotimes_{n=0}^{k} \mathbb{F}_2\{1, x^{2^n}\}$ to $\mathbb{F}_2[x]$ by first sending it to $\bigotimes_k \mathbb{F}_2[x]$ then to $\mathbb{F}_2[x]$. In particular, note that the colimit of the $\bigotimes_{n=0}^{k} \mathbb{F}_2\{1, x^{2^n}\}$, w.r.t. k, is isomorphic to $\mathbb{F}_2[x]$. Let $e_n = x^{2^n}$ and let $e_I = e_{i_1} \ldots e_{i_k}$ for $I = (i_1, \ldots, i_k)$, then e_I form a basis of $\mathbb{F}_2[x]$. Now, notice that the complex C_* is the tensor product of the $D_n (n \geq 0)$, where $D_n = \mathbb{F}_2\{1, e_n\} \otimes \mathbb{F}_2[y_{2^n+1}]$. where $de_n = y_{2^n+1}$, $dy^q_{2^n+1} = 0$, $dy^q_{2^n+1}e_n = y^q_{2^n+1}de_n$. Then by explicit computation, we find that the homology of D_n is $\mathbb{Z}_2$ at 0 and 0 at higher levels. And as we tensor these D_n together, we see that $H_0(C_*) = \mathbb{Z}_2$ and 0 at higher levels. This construction is often known as the koszul resolution.

So back to our old discussion, why is x transgressive? Steenrod Algebra! Note that $x \in H^n \implies Sq^n(x) = x^2$. We have the Kudo transgression theorem which works for any SSS: suppose we have $F \to E \to B$ and suppose $\pi_1 B$ acts trivially. If $x \in H^j(x)$ is transgressive, then $Sq^k(x)$ is transgressive and the transgression of $Sq^k(x)$ is the Sq^k of the transgression of x.

Proof. Start by considering the definition of transgression:

$$\begin{tikzcd} H^{j+1}(E,F) & H^j(F) \arrow[l, "\delta"'] \\ H^{j+1}(B) \arrow[u, "p^*"] & \end{tikzcd}$$

to say that γ is the tresgression of x is to say that $p^*(y) = \delta(x)$. Now apply Sq^k to everything, and it suffices to show that the diagram still commutes.

We need $Sq^k(\delta x) = \delta(Sq^k x)$. Consider the definition of the connecting homomorphism. The following are all isomorphisms in cohomology:

$$\begin{tikzcd} (E \cup CF, CF) & (E \cup CF, *) \arrow[l] \\ (E,F) \arrow[u] & \end{tikzcd}$$

And we see there's a mapping $E \cup CF \to \Sigma F$, which then induces a mapping $H^{k+1}(E \cup CF) = H^{k+1}(E,F) \leftarrow H^{k+1}(\Sigma F) \cong H^k(F)$. So the connecting homomorphism is just the suspension isomorphism followed by a map. Then it is clear that this connecting homomorphism commutes with Sq^k. Finally, it is a straightforward check that the diagram above still works after Sq^k.

□

Now using the Kudo regression theorem, we know that $x^2 = Sq^1(x)$ transgresses to $Sq^1(y_2) = y_3$. Similarly, $x^2 = Sq^2(x^2)$ transgresses to $Sq^(y_3) = y_5$. Therefore we have:

Corollary 9. $H^*(K(\mathbb{Z}_2, 2)) = \mathbb{Z}_2[y, Sq^1 y, Sq^2 Sq^1 y, Sq^4 Sq^2 Sq^1 y, \ldots]$.

Now consider the same settting but applied to $K(\mathbb{Z}_2, 2) \to * \to K(\mathbb{Z}_2, 3)$. By Borel's theorem, we can conclude that $K(\mathbb{Z}_2, 3)$ again has a polynomial algebra cohomology. In particular, this allows us to compute the cohomology of all $K(\mathbb{Z}_2, n)$. Next time we'll introduce a nice way to clean things up.

21 Computing with Steenrod Squares (Guest Lecturer: Jeremy Hahn)

Recall from the last lecture that we have the natural operations $Sq^k : H^k(\bullet, \mathbb{F}_2) \to H^{*+k}(\bullet, \mathbb{F}_2)$ satisfying the Steenrod axioms.

Definition 21. *For any sequence $I = (i_m, i_{m-1}, \ldots, i_1)$, we use Sq^I to denote the composition $Sq^{i_m} \circ \ldots \circ Sq^{i_1}$.*

Last time we used Serre Spectral Sequence to compute $H^k(K(\mathbb{F}_2, n))$.

Definition 22. *A sequence $I = (i_m, \ldots, i_1)$ is admissible if $i_k \geq 2i_{k-1}$ for all k. The excess of I is then defined as $(i_m - 2i_{m-1}) + (i_{m-1} - 2i_{m-2}) + \ldots + (i_2 - 2i_1) + i_1$.*

Theorem 21.1. *The cohomology $H^*(K(\mathbb{F}_2, n)) = \mathbb{F}_2[Sq^I \iota_n]$ where I runs over all admissible sequences of excess $< n$*[9]*.*

21.1 Cohomology Operations

Definition 23. *Fix an integer l, a cohomology operation of degree i is a natural transformation $H^l(\bullet, \mathbb{F}_2) \to H^{l+i}(\bullet, \mathbb{F}_2)$.*

Examples: in degree 5, we have $Sq^5, Sq^2Sq^3 = Sq^{(2,3)}, Sq^5 + Sq^{(2,3)}$, etc.

Definition 24. *A **stable** cohomology operations of degree i is a natural transformation $H^*(\bullet, \mathbb{F}_2) \to H^{*+i}(\bullet, \mathbb{F}_2)$ commuting with the suspension isomorphism, i.e. $H^*(X) \cong H^{*+1}(\Sigma X)$.*

Note that every cohomology operation can be extended to a stable cohomology operation in our current setting, so strictly speaking we don't have to worry about the term "stable" that much.

Remark 7. *$H^l(\bullet, \mathbb{F}_2)$ is represented by $K(\mathbb{F}_2, l)$. By Yoneda's lemma, cohomology operations are in bijection with $H^{l+i}(K(\mathbb{F}_2, l))$, and these are sums of Sq^I by what we said above.*

Remark 8. *All apparently "non-admissible" operations are in fact admissible, e.g. $Sq^{(2,3)} = Sq^5 + Sq^{(4,1)}$.*

So this brings in the question: how can we prove that two stable cohomology operations are the same?

Theorem 21.2. *Suppose a, b are both degree i stable cohomology operations, and suppose that they agree on products of 1-dimensional classes. Then a and b are the same stable cohomology operation.*

Corollary 10. *Sq^i is the unique degree i stable cohomology operations satisfying:*

1. *The Cartan formula, and*
2. *$Sq^0(x) = x, Sq^1(x) = x^2, Sq^i(x) = 0$ for $i \geq 2$ when x is a 1-dimensional class.*

Let's see a few examples on how to use this theorem first.

Example 11. $Sq^1Sq^1 = Sq^{(1,1)} = 0$.

Proof. If x is one dimensional, then $Sq^1(Sq^1(x)) = Sq^1(x^2) = xSq^1(x) + Sq^1(x)x = 2xSq^1(x) = 0$. Otherwise suppose $x = yz$, where y, z are smaller products of 1-dimensional classes. Then $Sq^1(Sq^1(yz)) = Sq^1(Sq^1(y)z + ySq^1(z)) = Sq^{(1,1)}(y)z + Sq^1(y)Sq^1(z) + Sq^1(y)Sq^1(z) + ySq^{(1,1)}z = 0$ by induction. □

Example 12. $Sq^1Sq^2 = Sq^3$. *(Note that the RHS is admissible.)*

Proof. If x is a 1-dimensional class then both sides are 0. Now suppose $x = yz$, then $Sq^3(yz) = Sq^3(y)z + Sq^2(y)Sq^1(z) + Sq^1(y)Sq^2(z) + ySq^3(z) = Sq^{(1,2)}(y)z + Sq^2(y)Sq^1(z) + Sq^1(y)Sq^2(z) + ySq^{(1,2)}z$ (by induction). On the other hand,$Sq^{(1,2)}(yz) = Sq^1(Sq^2(yz)) = Sq^1(Sq^2(y)z + Sq^1(y)Sq^1(z) + ySq^2(z)) = Sq^{(1,2)}(y)z + Sq^2(y)Sq^1(z) + Sq^{(1,1)}(y)Sq^1(z) + Sq^1(y)Sq^{(1,1)}(z) + Sq^1(y)Sq^2(z) + ySq^{(1,2)}(z)$, which is what we had before since $Sq^{(1,1)} = 0$. □

Proof of the Theorem. We first need a lemma. Consider $x_1 \ldots x_n \in H^n(\underbrace{\mathbb{RP}^\infty \times \ldots \times \mathbb{RP}^\infty}_{\text{n copies}}) = \mathbb{F}_2[x_1, \ldots, x_n]$; by naturality, it corresponds to a map $(x_1 \ldots x_n)^* : \underbrace{\mathbb{RP}^\infty \times \ldots \times \mathbb{RP}^\infty}_{\text{n copies}} \to K(\mathbb{F}_2, n)$.

[9] Mike's note says $\leq n$, but that might have been a mistake.

Lemma 11. *The natural map* $(x_1 \dots x_n)^*$ *produces a monomorphism in* $H^l(\bullet, \mathbb{F}_2)$ *for all* $l \leq 2n$.

Proof of the Lemma. We'd like to show that $H^l(K(\mathbb{F}_2, n)) \hookrightarrow H^l(\mathbb{RP}^{\infty \times n})$ for $l \leq 2n$. For this, we want to show that the images of $Sq^I(\iota_n)$ are linear independent for I admissible of total degree $\leq 2n$. The images are $Sq^I(x_1 \dots x_n)$. Each of these will be symmetric polynomials in the x_i, so we can express them in terms of elementary symmetric polynomials ($\sigma_1 = x_1 + \dots + x_n, \sigma_2, \dots$). We want to calculate $Sq^I(\sigma_n)$. The rest is just calculation, and follow the claims below.

Claim 1 $Sq^i(\sigma_n) = \sigma_n \sigma_i$.

Claim 2 $Sq^I(\sigma_n) = \sigma_n \sigma_{i_m} \dots \sigma_{i_1} +$(sum of monomials of smaller lexigraphical order) where $I = (i_m, \dots, i_1)$. □

Now we want to show that $a, b : H^n(\bullet) \to H^{n+i}(\bullet)$ are the same for all n. Of course, this is saying that a and b define the same element in $H^{n+i}(K(\mathbb{F}_2, n))$, which is the same to say that $a(\iota_n) = b(\iota_n)$ for all n. By stability, it suffices to prove the statement for $n > i$. By the lemma, it suffices to show that $a(x_1 \dots x_n) = b(x_1 \dots x_n)$ (of course we're abusing the notations a bit). But we have already assumed that a and b agree on products of 1-dimensional classes. □

Some side notes: Steenrod squares are examples of *primary cohomology operations*; it turns out that there are also higher cohomology operations, which one discovers from the study of Adam spectral sequences. It turns out that though the primary cohomology operations do not completely capture the structure of the cohomology, along with the higher cohomology operations they do; in other words, two cohomology theories that agree on all cohomology operations would be equal to each other.

22 Adem Relation and Bocksteins (Guest Lecturer: Xiaolin Shi)

Recall that we have proven the axiomatized description of the Steenrod operations and the uniqueness theorem. Let's just restate it here:

Theorem 22.1 (Uniqueness)**.** *The Steenrod squares are the unique cohomological operations such that:*

- $Sq^0(x) = 0$.
- $Sq^{|x|}(x) = x^2$.
- $Sq^k(x) = 0$ *for* $k > |x|$.
- Sq^k *commutes with the suspension isomorphism.*
- *Cartan formula holds.*

We also briefly touched on–though did not prove–the Adem relation, which allows us to write an arbitrary Steenrod product to an admissible one.

22.1 Adem Relation

Proposition 17 (Adem Relation)**.** $Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$.

Keep in mind that everything is mod 2.

Example 13. $Sq^3 = Sq^1 Sq^2$ *(we have shown this last time).*

Example 14. $Sq^2 Sq^3 = \binom{3-0-1}{2-2\cdot 0} Sq^5 Sq^0 + \binom{3-1-1}{2-2\cdot 1} Sq^4 Sq^1 = Sq^5 + Sq^4 Sq^1 \implies Sq^5 = Sq^2 Sq^3 + Sq^4 Sq^1$ *(since we work mod 2).*

Today we talk about the Bockstein homomorphisms.

22.2 Bockstein Homomorphisms

We already know that $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2) = \mathbb{Z}_2[Sq^I \iota_n]$ for admissible I with excess less than n. We computed this via the Serre spectral sequence. By the same method, one can compute $H^*(K(\mathbb{Z}, n); \mathbb{Z}_2) = \mathbb{Z}_2[Sq^J \iota_n]$ with admissible J with excess less than n, and that if $J = (j_m, \dots, j_1)$, then $j_1 > 1$. Today we'd like to compute $H^*(K(\mathbb{Z}/2_k, n); \mathbb{Z}_2)$ for $k > 1$ based on the facts above. (From now on we drop the $\mathbb{Z}_2$ coefficient.)

First step, let's consider $n = 1$. Consider the SES $0 \to \mathbb{Z} \xrightarrow{2^k} \mathbb{Z} \to \mathbb{Z}/2^k \to 0$. We get a fibration $K(\mathbb{Z}, 1) \xrightarrow{2^k} K(\mathbb{Z}, 1) \to K(\mathbb{Z}/2^k, 1)$[10]. We know that $K(\mathbb{Z}, 1) = S^1$. Now we run the Serre spectral sequence on this fibration: $E_2^{p,q} = H^p(K(\mathbb{Z}/2^k, 1); H^q(S^1)) \Rightarrow H^{p+q}(S^1)$. This means we have to only consider two rows.[11]

4							
2							
	$\mathbb{Z}_2(a)$	$\mathbb{Z}_2(a\iota_1)$	$\mathbb{Z}_2(a\beta_k)$	$\mathbb{Z}_2(a\iota_1\beta_k)$	$\mathbb{Z}_2(a\beta_k^2)$	$\mathbb{Z}_2(a\iota_1\beta_k^2)$	$\mathbb{Z}_2(a\beta_k^3)$
0	$\mathbb{Z}_2(1)$	$\mathbb{Z}_2(\iota_1)$	$\mathbb{Z}_2(\beta_k)$	$\mathbb{Z}_2(\iota_1\beta_k)$	$\mathbb{Z}_2(\beta_k^2)$	$\mathbb{Z}_2(\iota_1\beta_k^2)$	$\mathbb{Z}_2(\beta_k^3)$
	0		2		4		6

[10] How do we know this is a simple fibration?

[11] I'm using the sseq package for spectral sequences from now on.

Both rows are $(H^0, H^1, \ldots)$, and we know that $H^0 = H^1 = \mathbb{Z}_2$ since we're with $\mathbb{Z}_2$ coefficients. On E_∞ we should have $\mathbb{Z}_2$ for $H^1(E)$, so $(0,1)$ must have been killed, and thus the first nontrivial differential is a monomorphism, and because $(2,0)$ must be killed we know that that particular differential is in fact an isomorphism, and thus $H^2 = \mathbb{Z}_2$. Continue this process to the right, we see that $H^j = \mathbb{Z}_2$ for every j.

Now let us assign names to the generators as indicated in the graph above (note that ι_1 is the fundamental class of $K(\mathbb{Z}_{2^k}, 1))$, and compute the rest using the Leibniz rule. Then we see that $H^*(K(\mathbb{Z}/2^k, 1) = \mathbb{Z}_2[\iota_1, \beta_k]/(\iota_1^2 = 0)$.[12] This yields that $H^*(K(\mathbb{Z}/2^k, 1) = \mathbb{Z}_2[\iota_1, \beta_k]/(\iota_1^2 = 0) = \mathbb{Z}_2[\iota_1]/(\iota_1^2 = 0) \otimes \mathbb{Z}_2[\beta_k] = H^*(K(\mathbb{Z}, 1)) \otimes H^*(K(\mathbb{Z}, 2))$.

In general, this calculation yields that $H^*(K(\mathbb{Z}/2^k, n); \mathbb{Z}_2) = H^*(K(\mathbb{Z}, n)) \otimes H^*(K(\mathbb{Z}, n+1))$, which one can write as $\mathbb{Z}_2[Sq^I \iota_n, Sq^J \beta_k]$. One can then consider the pullback fibration $K(\mathbb{Z}/2^k, n) \to K(\mathbb{Z}, n+1) \xrightarrow{2^k} K(\mathbb{Z}, n+1)$, then the $\beta_k \in H^{n+1}(K(\mathbb{Z}/2^k, n))$ is the restriction from $\iota_{n+1} \in H^{n+1}(K(\mathbb{Z}, n+1))$, so we have $\beta_k \in [K(\mathbb{Z}/2^k, n), K(\mathbb{Z}/2, n+1)]$. This yields a stable natural transformation $\beta_k : H^n(X; \mathbb{Z}/2^k) \to H^{n+1}(X; \mathbb{Z}_2)$.

But where do we get this β_k, really? Consider the fibration $\mathbb{Z}_2 \to \mathbb{Z}/2^{k+1} \to \mathbb{Z}/2^k$, and consider its associated fiber sequence:

$$\ldots \to H^n(X; \mathbb{Z}_2) \to H^n(X; \mathbb{Z}/2^{k+1}) \to H^n(X; \mathbb{Z}/2^k) \xrightarrow{\beta_k} H^{n+1}(X; \mathbb{Z}_2) \to \ldots$$

In the special case of $k = 1$, we have $\beta_1 : H^n(X; \mathbb{Z}_2) \to H^{n+1}(X; \mathbb{Z}_2)$, which coincides with Sq^1.

[12] How do we know that $\beta_k \neq \iota_1^2$? We can compare its action as a cohomological operation with ι_1^2.

23 More on Adem Relation

Again, welcome back Mike. Today we would like to prove the Adem relation.

23.1 Some Tricks for Computation

First, a fast trick to compute binomial coefficients mod 2. The validity of the following method can be easily proven.

Proposition 18. *Suppose p is a prime and $m = (a_1 \dots a_r)_p$, $n = (b_1 \dots b_r)_p$, then* $\binom{m}{n} = \prod_{i=1}^{r} \binom{a_r}{b_r} \pmod p$.

Example 15. $\binom{5}{4} = \binom{(101)_2}{(100)_2} = \binom{1}{1}\binom{0}{0}\binom{1}{0} = 1 \pmod 2$.

Second, recall that for $x \in H^d(X)$, we the total power operation $P(x) \in H^{2d}(X \times \mathbb{RP}^\infty) = Sq^d(x) + Sq^{d-1}(x)t + \ldots + Sq^0(x)t^d \in H^*(X)[t]$, where the Steenrod operations arise as coefficients of this polynomial. In order to make bookkeeping easier, another way of writing this is by the lower index: $Sq_0(x) + Sq_1(x)t + \ldots + Sq_d(x)t^d$, where $Sq_{d-i}(x) = Sq^i(x)$.

23.2 A New Treatment

We can also write the Adem relation in the lower-index form. Suppose we apply P again. For the sake of clarity, let's write the $P(x)$ above as $P_t(x)$. So we have $P_t(x) = \sum_i Sq_i(x)t^i$. Now consider the formal operator $L(t) = \sum_i Sq_i t^i$, then $L(s) \cdot L(t) = \sum_{i,j} Sq_j Sq_i s^i t^j$, which we can consider as a generating function for $Sq_j Sq_i$. We have proven before that P_t is a ring homomorphism, which yields additivity and the Cartan formula. One more detail for calculation: suppose $X = \mathbb{RP}^\infty$, and $0 \neq x \in H^1(\mathbb{RP}^\infty)$, then $P_t(x) = x^2 + xt$.

Now suppose $x \in H^d(X)$. Let us calculate $P_t(P_s(x))$. It equals $P_t(\sum_i Sq_i(x)s^i) = \sum_i P_t(Sq_i(x))P_t(s^i) = \sum_i \sum_j Sq_j Sq_i(x)t^j P_t(s)^i = \sum_i \sum_j Sq_j Sq_i(x)t^j(s^2+st)^i = L(t) \cdot L(s^2+st)$. Now we shall prove that that $L(s) \cdot L(s^2 + st)$ is symmetric in s and t, and thus we have **the Adem Relation**:

$$L(t) \cdot L(s^2 + st) = L(s) \cdot L(t^2 + st).$$

Symmetric Property. We need to show that $P_t \circ P_s(x) = P_s \circ P_t(x)$ for x being a product of 1-dimensional classes (then invoke the uniqueness theorem of stable cohomology operations). Since both sides are ring homomorphisms, it suffices to do the case in which x is a 1-dimensional class. Now, $P_t \circ P_s(x) = P_t(x^2 + sx) = (x^2 + xt)^2 + (s^2 + st)(x^2 + xt) = x^4 + (t^2 + s^2 + st)x^2 + (s^2t + st^2)x$, which is symmetric. □

(We'll show later that this symmetric property actually stems from the very definition of things, and don't really rely on the lucky calculation above.)

Now we show that the form above of the Adem relation coincides with the usual one. We want

$$\sum_{i,j} (s^2 + st)^i t^j Sq_i Sq_j = \sum_{i,j} (t^2 + st)^i s^j Sq_i Sq_j.$$

Let $u = s^2 + st$, then the left side becomes $\sum_{i,j} u^i t^j Sq_i Sq_j$. How do we handle the right side? Use residues. Suppose we have a formal Laurent series $f(x) = \sum_{i > -\infty} a_i x^i$, then $\mathrm{Res}_{x=0} f(x)dx = a_{-1}$, which is the coefficient of $\frac{dx}{x}$. Also, suppose $x = g(y), g(0) = 0, g'(0)$ is nonzero, so g^{-1} exists by the inverse function theorem.

Theorem 23.1. *$Res_{x=0} f(x)dx = Res_{y=0} f(g(y))g'(y)dy$.*

Proof. See any complex analysis textbook. □

Now look at $\sum_{i,j} u^i t^j Sq_i Sq_j$, so $\sum_n t^n Sq_n Sq_m$ is the coefficient of u^m in it.[13] Now $u = s^2 + st, du = tds$ (where s is the variable), thus we know that

$$\sum_n t^n Sq_n Sq_m = \text{coefficient of } u^m \text{ in } \sum_{i,j} u^i t^j Sq_i Sq_j = \mathrm{Res}_{u=0} \frac{u^i s^j Sq_j Sq_i}{u^{m+1}} du$$

which further simplifies to

$$\mathrm{Res}_{s=0} \frac{t^{i+1}(s+t)^i s^j}{s^{m+1}(s+t)^{m+1}} Sq_i Sq_j ds = \mathrm{Res}_{s=0} \frac{t^{i+1}(s+t)^{i-m-1} s^j}{s^{m+1}} ds Sq_i Sq_j$$

which is the coefficient of s^m in $\sum_{i,j} t^{i+1}(s+t)^{i-m-1} s^j Sq_i Sq_j$, thus $t^n Sq_n Sq_m = \binom{i-m-1}{m-j} t^{2i+j-2m} Sq_j Sq_i$, from which it is immediate by a substitution that

$$Sq_n Sq_m = \binom{i-m-1}{2i-m-n} Sq_{2m+n-2i} Sq_i.$$

[13] Sorry for the notation mess.

24 Even More on Adem Relation

Let's first investigate a more elementary way to prove the relation $P_s(P_t(x)) = P_t(P_s(x))$. Recall the diagram:

$$\begin{tikzcd} \mathbb{RP}^\infty \times X \arrow[r] \arrow[dr, "P_t(x)"'] & S^\infty \times_{\mathbb{Z}_2} (X \times X) \arrow[d, "\tilde{P}_t(x)"] \\ & K(\mathbb{Z}_2, 2n) \end{tikzcd}$$

If we apply this action again, we get

$$\begin{tikzcd} \mathbb{RP}^\infty \times \mathbb{RP}^\infty \times X \arrow[r] \arrow[dr, "P_s(P_t(x))"'] & S^\infty \times_{\mathbb{Z}_2} (S^\infty \times_{\mathbb{Z}_2} (X \times X))^2 \arrow[d, "\tilde{P}_s(\tilde{P}_t(x))"] \\ & K(\mathbb{Z}_2, 4n) \end{tikzcd}$$

Consider the space $S^\infty \times (S^\infty \times S^\infty \times X^2 \times X^2)$ that "covers" the space on top right. There's one $\mathbb{Z}_2$ acting on the outside, and there's $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting on the inside, and the outer $\mathbb{Z}_2$ acts on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ by exchanging the factors. So we can write the space $S^\infty \times_{\mathbb{Z}_2} (S^\infty \times_{\mathbb{Z}_2} (X \times X))^2$ as $(S^\infty)^3 \times_{\mathbb{Z}_2 \wr \mathbb{Z}_2} X^4 \to K(\mathbb{Z}_2, 4n)$, where $\wr$ denotes the wreath product.

As a motivating example, suppose X is a point. The fundamental group of $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \times X$ in this case is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the fundamental group of $(S^\infty)^3/(\mathbb{Z}_2 \wr \mathbb{Z}_2)$ is $\mathbb{Z}_2 \wr \mathbb{Z}_2$, so the trivial mapping $\mathbb{Z}_2 \times \mathbb{Z}_2 \to *$, induced by $P_s \circ P_t$, actually factors as $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \wr \mathbb{Z}_2 \to *$. So what is this embedding? Note that by definition, $\Sigma_n \wr \Sigma_k \subseteq \Sigma_{nk}$, so $\mathbb{Z}_2 \wr \mathbb{Z}_2 \subseteq \Sigma_4$ acts on four elements (namely two copies of the former $\mathbb{Z}_2$). In particular, if the two copies are $\{1, 2\}$ and $\{3, 4\}$ respectively, then the generator of the first $\mathbb{Z}_2$ (the "outer" $\mathbb{Z}_2$) embeds as $(13)(24)$, and the generator of the second $\mathbb{Z}_2$ (the "inner" $\mathbb{Z}_2$) embeds as $(12)(34)$. One should note that these two actions are conjugate in Σ_4 by the element (23), as this conjugation relation is really why Adem relation would hold.

24.1 Higher Power Operations

It is easy to observe that $S^\infty \cong \{(x, y) \in \mathbb{R}^\infty \mid x \neq y\}$[14]. To generalize this, let $\tilde{C}_k(\mathbb{R}^\infty)$ denote the kth configuration spaces, i.e. $\{(x_1, \ldots, x_k) \in \mathbb{R}^\infty \mid x_i \neq x_j\}$. The configuration spaces are contractible spaces with free Σ_k actions. It turns out that $B\Sigma_k = \tilde{C}_k(\mathbb{R}^\infty)/\Sigma_k$ (modding out the natural action of Σ_k, i.e. taking unordered size-k sets). Define the **kth extended power** of $X \xrightarrow{x} K(\mathbb{Z}_2, n)$ as the unique extension to $\tilde{P}_k(x)$ (which we omit the definition) such that the following graph commutes and that $P_k(x) \circ i = x^k$:

$$\begin{tikzcd} X \arrow[r, "i"] & B\Sigma_k \times X \arrow[r] \arrow[dr, "P_k(x)"'] & \tilde{C}_k(\mathbb{R}^\infty) \times_{\Sigma_k} X^k \arrow[d, "\tilde{P}_k(x)"] \\ & & K(\mathbb{Z}_2, nk) \end{tikzcd}$$

The fact giving the Adem relation is that $P_2 \circ P_2$ factors through a total 4th power operation i.e. the map in fact factors as follows:

$$B\Sigma_2 \times B\Sigma_2 \times X = B(\Sigma_2 \times \Sigma_2) \times X \to \tilde{C}_2(\mathbb{R}^\infty)^3 \times_{\mathbb{Z}_2 \wr \mathbb{Z}_2} X^4 \to \tilde{C}_4(\mathbb{R}^\infty) \times_{\mathbb{Z} \wr \mathbb{Z}_2} X^4 \to \tilde{C}_4(\mathbb{R}^\infty) \times_{\Sigma_4} X^4 \to K(\mathbb{Z}_2, 4n)$$

where the only not-so-canonical map $\tilde{C}_2(\mathbb{R}^\infty)^3 \times_{\mathbb{Z}_2 \wr \mathbb{Z}_2} X^4 \to \tilde{C}_4(\mathbb{R}^\infty) \times_{\mathbb{Z}_2 \wr \mathbb{Z}_2} X^4$ comes from the following construction of $\tilde{C}_2{}^3 \to \tilde{C}_4$: the elements in the two "inner" $\tilde{C}_2$s are two pairs of points $(a_1, a_2), (b_1, b_2)$, such that $a_1 \neq a_2, b_1 \neq b_2$, and that each pair is indexed by by a pair of points in X; the outer $\tilde{C}_2$ then has two points $c_a \neq c_b$, indexed by the two inner pairs repsectively. We would like to consider only the two inner pairs, but of course they might collide; so what we need to do is use c_a and c_b to send them to two different directions so that they are apart. More precisely, consider the continuous mapping: let the "ball" containing a_1 and a_2 be of radius r_a, and the other "ball" containing b_1 and b_2 be of radius r_b, then send the two "balls" along the directions c_1 and c_2 by some large value depending on $\max(r_a, r_b)$ and $\cos(\theta)$ where θ is the angle between c_a and c_b, such that the center of the two balls are $3 \max(r_a, r_b)$ apart, meaning that they are disjoint, so $(a_1, a_2, b_1, b_2) \in \tilde{C}_4$ now.

[14] The latter deformation retracts to the former, as one can visually see from the finite cases.

Then, due to the following fact and the obvious fact that $\tilde{C}_4(\mathbb{R}^\infty) \times_{\Sigma_4} X^4 = B\Sigma_4 \times X$, the two maps under consideration are homotopic in their $B\Sigma_2 \times B\Sigma_2 \times X \to B\Sigma_4 \times X$ part, since we mentioned in the last section that the induced actions are conjugate. Since $\tilde{P}_4$ is uniquely defined, the second part $B\Sigma_4 \times X \to K(\mathbb{Z}_2, 4n)$ is unique, so we have shown that the two maps are homotopic.

Proposition 19. *Suppose we have $f_1, f_2 : H \to G$ are conjugate, i.e. $\exists g \in G.gf_1g^{-1} = f_2$, then $Bf_1, Bf_2 : BH \to BG$ are homotopic.*

Proof. Skipped. □

It remains to show that the map factors. In fact, we will show the stronger statement that $H^{4n}(\tilde{C}_4(\mathbb{R}^\infty) \times_{\mathbb{Z} \wr \mathbb{Z}_2} X^4) = H^{4n}(\tilde{C}_4(\mathbb{R}^\infty) \times_{\Sigma_4} X^4)$. Without loss of generality we consider the universal case w.r.t. cohomology pullback, i.e. when $X = K(\mathbb{Z}_2, n)$ and $A = *$, so that $H^*(X, A) = 0$ for $* < n$ by Hurewicz. Suppose G is a group acting on $(X, A)^k$ for some k, and that S is a contractible space with free G action. Let $B = S/G$, then B is a classifying space of G, i.e. $\pi_1 B = G$, higher homotopy groups vanish, and S is the universal cover. Suppose E is an arbitrary space with G action. Then we have the following theorem:

Theorem 24.1. *If $H^*(E) = 0$ for $* < r$, then $H^r(S \times_G E) \to H^r(S \times E) \to H^r(E)$ is a monomorphism with image those elements invariant under G.*

Proof. Use Serre spectral sequence on the fiber bundle $E \to S \times_G E \to B$. Consider the SSS, which has two identical columns on the E_2 page. Everything is zero until row n, where the first element becomes $H^0(B, H^n(E))$. The action of the fundamental group is nontrivial so we have to consider local coefficients; but, as one observes from inspecting the chain complex, this action is just the action of G, so the 0th cohomology is just the invariants of $H^n(E)$ under G. Since nothing can ever get cancelled, this fully determines the SSS. So $H^n(S \times_G E) \to H^n(F)$ is injective by edge homomorphism. □

But back to our story. Specializing with $E = (K(\mathbb{Z}_2, n), *)^4$, $r = 4n$, $S = \tilde{C}_4(\mathbb{R}^\infty)$ and $G = \Sigma_4$ and $G = \Sigma_2 \wr \Sigma_2$ respectively, we see that $H^{4n}(\tilde{C}_4(\mathbb{R}^\infty) \times_{\mathbb{Z} \wr \mathbb{Z}_2} X^4)$ and $H^{4n}(\tilde{C}_4(\mathbb{R}^\infty) \times_{\Sigma_4} X^4)$ (or really, their images under monomorphism) are respectively just elements of $H^r(K(\mathbb{Z}_2, n), *)^{\otimes 4}$ invariant under $\Sigma_2 \wr \Sigma_2$ and Σ_4 respectively. However, note that in our case, $(X, A)^k = (K(\mathbb{Z}_2, n), *)^4 = K(Z_2, n)^{\wedge 4}$ as one can check from definition, so the target space is $H^{4n}(K(\mathbb{Z}_2, n), *)^{\wedge 4}) = \mathbb{Z}_2^{\wedge 4} = \mathbb{Z}_2$. It is clear that the invariant of this $\mathbb{Z}_2$ under both $\mathbb{Z}_2 \wr \mathbb{Z}_2$ and Σ_4 is $\mathbb{Z}_2$ itself. Hence the isomorphism.

Moral of the story: Adem relation holds because something is invariant under Σ_4, not just $\mathbb{Z}_2 \wr \mathbb{Z}_2$. As an ending note, there is yet another pure homological-theoretical explanation to this story that involves explicitly computing out all cohomologies involved. The takeaway from that story is that the only possible obstruction to the Eilenberg-Maclane structure from having a E_∞ spectra is the higher homotopy groups of E-M spaces, which of course don't exist. (Charles: I might sketch out that construction in my blog. The reference here is Steenrod and Epstein, *Cohomology Operations*.)

24.2 Application of Steenrod Algebra: Homotopy Groups of Spheres

Let C_2 be the Serre class consisting of group A such that $nA = 0$ for $n \gg 0$ and n odd. (modding out C_2 is equivalent to ignoring all odd torsions.) We shall calculate $\pi_*(S^n)$ localized at 2 by replacing S^n by an easier space which is an equivalence mod C_2 through a range.

Lemma 12. *Suppose X, Y have finitely generated homology and $X \to Y$ is an isomorphism in $H_i(X; \mathbb{Z}_2)$ for $i < n$ and epi in $i = n$. Then $H_i(X; \mathbb{Z}) \to H_i(Y; \mathbb{Z})$ is iso mod C_2 for $i < n$ and epi mod C_2 for $i = n$.*

Proof. Easy by the UCT. □

Remark 9. *An equivalent condition is that $H^i(X; \mathbb{Z}_2) \longleftarrow H^i(Y; \mathbb{Z}_2)$ is iso for $i < n$ and epi for $i = n$.*

What we can say from the conditions above is that $\pi_i(X) = \pi_i(Y)$ mod C_2 for $i < n$. Thus in particular, if we can build spaces that cohomologically approximate S^n mod C_2, then we can us them to compute the homotopy groups of S^n mod C_2.

Take $S^n \to K(\mathbb{Z}, n)$. We know $H^*(K(\mathbb{Z}, n)) = \mathbb{Z}_2[Sq^I \iota_n]$, and more explicitly for the latter space, $H^n = \mathbb{Z}_2(\iota_n)$ (single $\mathbb{Z}_2$ summand generated by ι_n), $H^{n+1} = 0$ (because I must have first term greater than 1), so up till here it agrees with $H^*(S^n)$; but the next one, generated by $Sq^2 \iota_n$, is probably different. So let's modify $K(\mathbb{Z}, n)$ so it has closer comohological behavior.

In particular, consider the following diagram, where $X_1 \to K(\mathbb{Z}, n) \xrightarrow{Sq^2} K(\mathbb{Z}_2, n+2)$ is a fibration (X_1 is the homotopy fiber) and $K(\mathbb{Z}_2, n+1) \to X_1 \to K(\mathbb{Z}, n)$ is its one-step backup:

$$
\begin{tikzcd}
K(\mathbb{Z}_2, n+1) \arrow[r] & X_1 \arrow[d] & \\
S^n \arrow[ur] \arrow[r] & K(\mathbb{Z}, n) \arrow[r, "Sq^2"] & K(\mathbb{Z}_2, n+2)
\end{tikzcd}
$$

Note that since $\pi_n(K(\mathbb{Z}_2, n+2)) = 0$, the mapping from $\pi_n(X_1) \to \pi_n(K(\mathbb{Z}, n))$ is surjective, so $S^n \to K(\mathbb{Z}, n)$ lifts to $S^n \to X_1$. Run Serre spectral sequence (diagram omitted), we see that ι_{n+1} hits $Sq^2\iota_n$ by transgression, and $Sq^1\iota_{n+1} = Sq^3\iota_n$ by transgression theorem, and thus $Sq^3\iota_{n+1} = Sq^2Sq^2\iota_n = Sq^3Sq^1\iota_n = 0$ by Adem relation. Similarly $Sq^2Sq^1\iota_{n+1}$ hits $Sq^5\iota_n$. Consider what's left in the cohomology of X_1, one can check that it's $\mathbb{Z}_2$ in dimension n, one in dimension $(n+3)$, 2 classes in $(n+4)$, etc. We'll continue this computation in the next lecture.

25 Some Nontrivial Stable Homotopy Computation

Professor Hopkins recently discovered a new puzzle website made by A. Watanuki, a graduate student at RIMS, that generates spectral sequence computation problems called *Subedoku*, which are presented in a similar style to the famous Sudoku puzzles. The website can be found here.

25.1 Killing Cohomology

Keep in mind that the overall goal of ours is to compute homotopy groups of spheres. Below we shall sketch the overall strategy.

Start with the mapping $S^n \to K(\mathbb{Z}, n) = X_0 \to K(\mathbb{Z}_2, n+2)$, and choose X_1 to be the homotopy fiber of $X_0 \to K(\mathbb{Z}_2, n+2)$. As we said in last class, there is a lifting $f : S^n \to X_1$. Now choose the smallest m such that $H^m f : H^m X_1 \to H^m S^n$ is not a monomorphism, and let A be the kernel of $H^m f$. Then we have the mapping $S^n \to X_1 \to K(A, m)$, and we can continue the construction by letting X_2 be the homotopy fiber of $X_1 \to K(A, m)$, etc. (That m actually increases is not completely trivial, but is not hard to observe from the SSS on the backup fibration, as we have seen in the last class.)

More generally, in order to compute the cohomology of some space W, start with a mapping into a product of E-M spaces that is an approximation of homotopy groups on low dimensions: $f : W \to \prod_i K(\pi_i(W), i) = X_0$, such that f is surjective in $H^*(\bullet; \mathbb{Z}_2)$. Let m be the smallest integer such that there is a nontrivial kernel in $H^m(X_0) \to H^m(W)$, and let A be this kernel. Then we have $W \to X_0 \to K(A, m)$, and we can continue this process. In the end $W \to X_\infty$ will be an isomorphism in $H^*(\bullet; \mathbb{Z}_2)$.

The last piece of the story, of course, is that $X_1, X_2, \ldots$ are now cohomological approximations, and thus homotopy approximations, of S^n mod C_2. The homotopy of each X_i, on the other hand, is easy to compute inductively thanks to the homotopy LES of fibrations.

Remark 10. *This technique is called "killing cohomology one at a time". Frank Adams later observed that if instead of doing this one at a time, we kill all cohomologies simultaneously, we can obtain even stronger results; this leads to the Adams spectral sequence.*

25.2 A Few Notes

Consider the fibration $X_{k+1} \to X_k \xrightarrow{a} K(\mathbb{Z}_2, n+1)$ and the backup $K(\mathbb{Z}_2, n) \to X_{k+1} \to X_k$. Consider the spectral sequence of the latter. On the base, a lies on dimension $n+1$ as the first nonzero element, and ι_n is the first nonzero element on the fiber column, on dimension n. Let $(1) = Sq^1 \iota_n$, then (1) may cause trouble for $H^*(X_{k+1})$ approximating $H^*(X_k)$, because it may not get killed. Of course, if it does, things work out rather nicely.

Example 16. *Start with* $* \to K(\mathbb{Z}_4, 2) \to K(\mathbb{Z}_2, 2)$*. Take the fiber* X_1*, do the SSS calculation on* $K(\mathbb{Z}_2, 1) \to X_1 \to K(\mathbb{Z}_4, 2)$ *to observe that* $X_1 = K(\mathbb{Z}_2, 2)$*; then do the same construction again, and observe that* $X_2 = *$*, so we have the following diagram:*

$$\begin{tikzcd}
 & * \arrow[d] & \\
K(\mathbb{Z}_1) \arrow[r] & K(\mathbb{Z}_2, 2) \arrow[r] \arrow[d] & K(\mathbb{Z}_2, 2) \\
* \arrow[r] \arrow[ur] & K(\mathbb{Z}_4, 2) \arrow[r] & K(\mathbb{Z}_2, 2)
\end{tikzcd}$$

which tells us that $H^*(*) = H^*(*)$*. Yeah.*

Example 17. *Consider* $K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_4, n) \to K(\mathbb{Z}_2, n)$ *and its SSS. We have* ι_n *and* $Sq^1 \iota_n$ *at* n *and* $n+1$ *respectively on the base, and* ι_n' *and* $Sq^1 \iota_n'$ *at* n *and* $n+1$ *on the fiber. The problem is that we have a mapping from* ι_n' *to* $Sq^1 \iota_n$*, and it's not clear immediately how this differential will work.*

In general, suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is the fibration, a nice graphical way to represent SSS with Bockstein homomorphisms is to connect u at $(n, 0)$ and u' at $(n+1, 0)$ with a r-fold line to indicate $\beta_r(u) = u'$ (which is a $H^n(X, \mathbb{Z}^{2^r}) \to H^{n+1}(X, \mathbb{Z})$, and connect v at $(0, m)$ and v' at $(0, m+1)$ with a s-fold line to indicate a $\beta_s(v) = v'$. Then, if there is a transgression $\tau(v) = u'$, then in the total homology of E there is a nontrivial Bockstein homomorphism β_{r+s} such that $i^* \beta^{r+s} p^*(u) = v'$. This is known as the **Bockstein Lemma** and detailed discussion can be found on page 106 of this book.

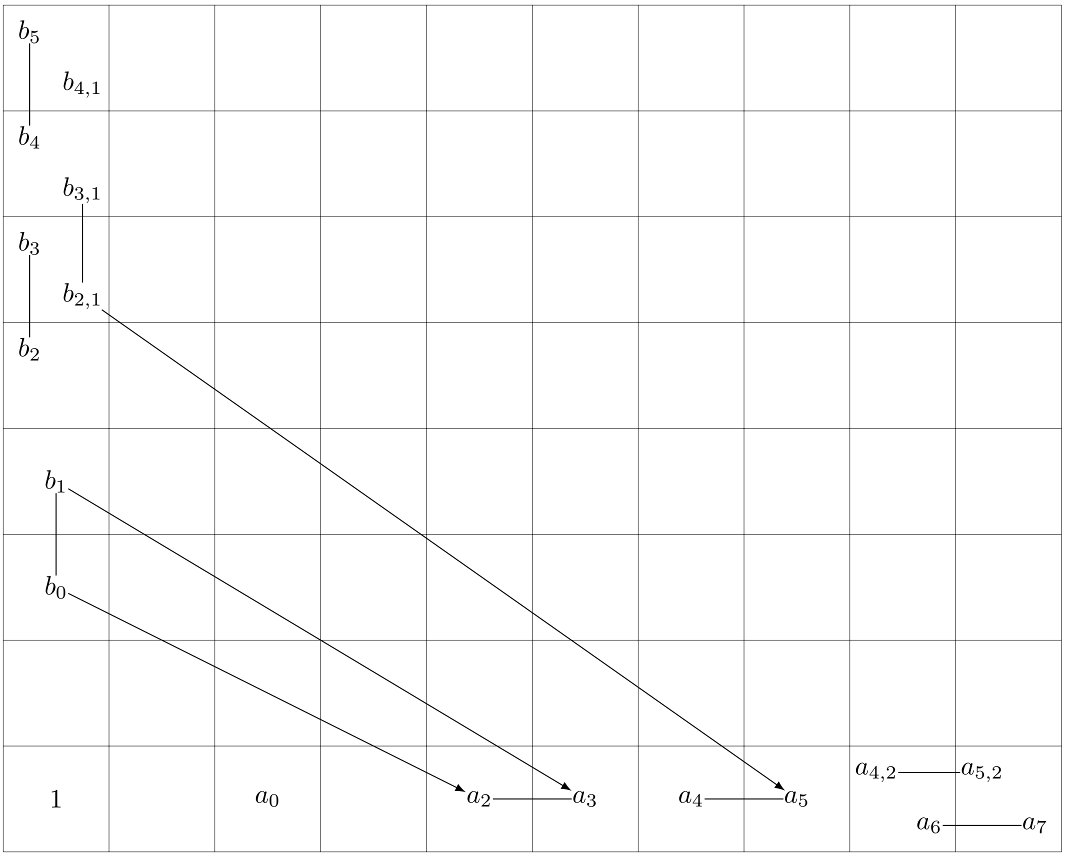
In general, this tells us that the problem of using the technique we have so far to compute all homotopy groups of spheres is that eventually we'll hit problem at dimension 15, where a class just "pops up" and we can't kill it with our current technique.

25.3 Back to Computation

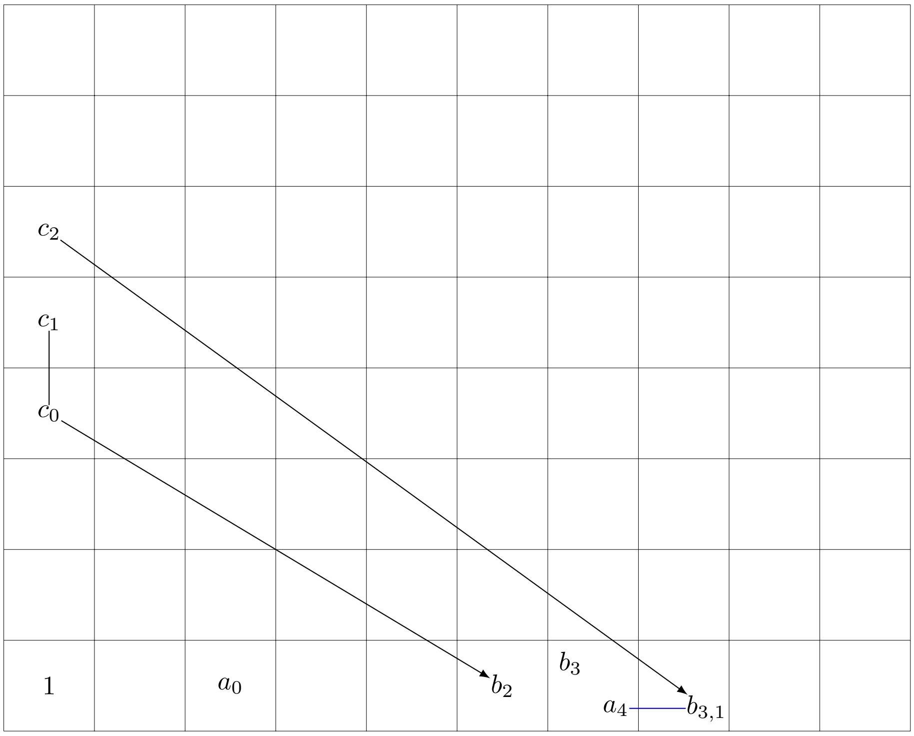
Recall that last time we went up to the second level of the tower:

$$\begin{tikzcd}
& K(\mathbb{Z}_2, n+1) \arrow[r] & X_1 \arrow[d] & \\
S^n \arrow[urr] \arrow[rr] & & K(\mathbb{Z}, n) \arrow[r, "Sq^2"] & K(\mathbb{Z}_2, n+2)
\end{tikzcd}$$

Now let $a_I = Sq^I \iota_n$ in $K(\mathbb{Z}, n)$, and let $b_I = Sq^I \iota_{n+1}$ in $K(\mathbb{Z}_2, n+1)$. Consider the SSS:



We know that b_0 goes to a_2, thus b_1 goes to a_3; further computations now call for Adem relations. b_2 and thus b_3 goes to zero because $db_2 = Sq^2 \iota_{n+1} = Sq^2 Sq^2 \iota_n = Sq^3 Sq^1 \iota_n = 0$. Some more computation shows that $b_{2,1}$ goes to a_5. But we have a pair $(a_4, b_{3,1})$ that is probably nontrivial; in fact, $a_4 \to b_{3,1}$ is a $\mathbb{Z}_4$ bockstein homomorphism on the kernel of Sq^1 (since $Sq^1 a_4 = a_5$ gets killed) by the Bockstein lemma above. This allows us to write down the cohomology of X_1 in a small range.



The next class we have to kill is b_2 (the package I'm using is unable to draw double arrow, so I'm using blue color instead). So we map $X_1 \to K(\mathbb{Z}_2, n+3)$ and take the fiber X_2, and back up to get $K(\mathbb{Z}_2, n+2) \to X_2$. Draw SSS again as in the graph above. c_0 hits b_2, c_1 hits b_3. c_2 hits $Sq^2 b_2 = b_{3,1}$, so we have the trouble with (a_4, c_3), the same situation again, which this time yields a $\mathbb{Z}_8$ Bockstein. Repeat this process again with $X_2 \to K(\mathbb{Z}_8, 4)$, we finally kill the pair, as one can check. Thus we have the following diagram so far:

$$\begin{tikzcd}
K(\mathbb{Z}_8, n+3) \arrow[r] & X_3 \arrow[d] & \\
K(\mathbb{Z}_2, n+2) \arrow[r] & X_2 \arrow[r] \arrow[d] & K(\mathbb{Z}_2, n+4) \\
K(\mathbb{Z}_2, n+1) \arrow[r] & X_1 \arrow[r] \arrow[d] & K(\mathbb{Z}_2, n+3) \\
S^n \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & K(\mathbb{Z}, n) \arrow[r, "Sq^2"] & K(\mathbb{Z}_2, n+2)
\end{tikzcd}$$

Thus we have the conclusion:

Theorem 25.1. *After localizing at 2, we have* $\pi_n S^n = \mathbb{Z}, \pi_{n+1} S^n = \mathbb{Z}_2, \pi_{n+2} S^n = \mathbb{Z}_2, \pi_{n+3} S^n = \mathbb{Z}_8$.

More detailed calculation for up to the 7th stable homotopy group is carried out in this book. But as we have seen above, this method has limitations (at dimension 15). As a matter of fact, when you run this on other localizations, e.g. localized at 3, you hit the same problem much more quickly.

Remark 11. *Next topic is spectra and cobordism. Review your smooth manifolds and transversality.*

26 Introducing Cobordism

26.1 Cobordism: An Overview

Recall the days of singular homology (back when AT was easy)? There's something closely related to that, which doesn't have an official name, but people call it (co)bordism homology,

Definition 25. *For X, let $\Omega_n(X)$ be the set of all maps from smooth n-manifolds M to X up to an equivalence relation known as cobordism.*

Definition 26. *Two maps $f_i : M_i \to X$, $i \in \{1, 2\}$, is said to be* ***cobordant*** *if there is some $h : N \to X$ where N is a smooth $(n+1)$-manifold, such that $\partial N = M_1 \coprod M_2$, and the restriction onto M_i of h is f_i.*

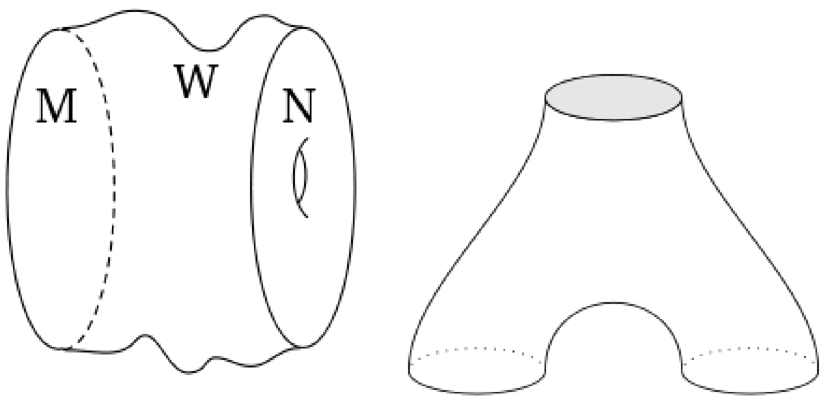


Figure 1: Illustrations from Wikipedia.

Of course, $\Omega_n(X)$ is a commutative monoid under disjoint union. But in fact it has an identity:

Lemma 13. *$\Omega_n(X)$ is a group. In particular, $M \coprod M$ is null-cobordant.*

Proof. Let $X = M \times I$, then X is a $(n+1)$-manifold with boundary $(M \coprod M) \coprod \emptyset$. □

For every $X \to Y$, we can define $\Omega(X) \to \Omega(Y)$, and we can in fact prove Mayer-Vietoris, but we'll skip the proof. In particular, let $f : M \to X = U \cup V$, and then $f^{-1}(U), f^{-1}(V)$ is a covering of M. By smooth partition of unity, one can choose a function that's 0 on one of them, 1 on the other of them, and in the intersection it's only increasing. Choose a submanifold L in the intersection and map it to a regular value, then we have $f|_L : L \to U \cup V$ is the connecting homomorphism in this homology.

In general, we know that $\Omega_n(X)$ is a generalized homology theory, where $\Omega_n(X, A)$ is defined as the set of cobordism classes of maps from pairs of smooth n-manifolds $(M, \partial M)$ to (X, A). We'll show this in a completely different approach from what we said above, namely the approach by Pontryagin and Thom, by identifying it with the homotopy group of something else (Thom spaces), and then compute them. (Don't worry, unlike spheres, they are easy to compute.)

The big question, of course, is what is $\Omega_n(X)$. But before that, what is $\Omega_n(*)$? We'll show that it is a ring under cartesian product that is isomorphic to $\mathbb{Z}_2[x_2, x_4, x_5, \dots \mid i \neq 2^k - 1]$, and then $\Omega_n(X)$ is just the regular homology under the coefficient group of this ring. From here on, many more stories unfold.

26.2 Vector Bundles

Informal Definition A vector bundle over S is (roughly) a family of vector spaces parametrized by S, i.e. for each $x \in S$, there should be a corresponding V_x vector space, such that the map is continuous in some obvious way.

There are two approaches to realize this idea. Suppose all of the V_x live in a huge fixed vector space W. On one hand, we can look at $Gr_n(W)$, the Grassmannian. (We computed its cohomology last term.) A vector bundle then is a continuous map $S \to Gr_n(W)$. The second approach is consider the maps $p : V \to S$, and take $V_x = p^{-1}(x)$, and somehow we need this to be a vector space. We mainly need some $+ : V \times_S V \to V$ that is commutative, associative, 0 being the unit, and for every $\lambda \in \mathbb{R}$, all operations are $\mathbb{R}$-homomorphisms.

Unfortunately, this naïve approach won't quite work because we can't guarantee *local triviality.* This condition says that for each $x \in S$ has a neighborhood $U \supseteq S$, such that V_U, being the pullback of $U \to S$

and $V \to S$, is isomorphic to $U \times \mathbb{R}^n$ and such that $V_U \to U$ is pullback of projection $U \times \mathbb{R}^n \to U$ by the isomorphism. (This notion was actually not clear for a while in history.) Once we add this condition in, things start to work out nicely.

We'll show in next lecture that over finite CW complex, every vector bundle is pullback over some *universal* vector bundle over the Grassmannian. For example, take the case of $\mathbb{R}^n$. There is a tautological vector bundle given by $V_k \to Gr_k(\mathbb{R}^n)$ such that the fiber over a subspace H of dimension k is just H. In other words, $V_k \subseteq Gr_k(\mathbb{R}^n) \times \mathbb{R}^n = \{(H, x) \mid x \in H\}$.

We need to check local triviality. Suppose H is a point in $Gr_k(\mathbb{R}^n)$, then we have its complement $H^\perp$ in $\mathbb{R}^n$. Then the needed $U = \{H' \subseteq \mathbb{R}^n \mid H' \cap H^\perp = 0\}$, i.e. spaces that projects to the entirity of H when projected by the projection $\pi : \mathbb{R}^n \to H$. On U, we can use π to identify $V|_U \cong U \times H$: $(H', x) \in V|_U \subseteq Gr_k(\mathbb{R}^n) \times \mathbb{R}^n \mapsto (H', \pi(x)) \in U \times H$.

Any operation on vector spaces should also translate to vector bundles. For example, $V \mapsto V^*$, $V, W \mapsto V \oplus W, V \otimes W$, $V \mapsto S^V$ (one-point compactification), $V \mapsto P(V)$ (projectivization), $V \mapsto Gr_k(V)$ (Grassmannian), etc. Notice that some of these produce topological spaces, not vector spaces, in which case we product fiber bundles. (How? we apply the operations fiber-wise, and check local nontriviality by checking that its result on the trivial (local) product bundle remains trivial.)

A remark: $V \mapsto S^V$ produces a fiber bundle over X. But note that now every fiber has a special point (the point at infinity), and it's a good idea to collapse all of these points to one. To do this, we instead use the Thom complex: $\mathrm{Thom}(X, V) = X^V = S^V/X$ (collapsing the infinity section). It turns out this construction has a lot of beautiful properties.

So, for the record, how would SSS of this fiber bundle look like? We'll have $H^*(X, H^*(S^n, *)) \Rightarrow H^*(S^V, X)$. But if we draw it out, we just have a row representing the shifted cohomology of the base. Disappointing? Not, because remember the Steenrod operations...

27 Universal Vector Bundle

27.1 Existence of universal vector bundle

Let $\mathrm{Vect}_k(X)$ be the set of vector bundles over X of dimension k, modulo the isomorphism relation.

Theorem 27.1 (Existence of Universal Bundle)**.** *If X is a compact Hausdorff space, then for $n \gg 0$ the map $[X, Gr_k(\mathbb{R}^n)] \to Vect_k(X)$ that sends f to the pullback bundle f^*V_k for some universal V_k, is an isomorphism.*

Recall that by the pullback we mean the following diagram:

$$\begin{tikzcd} f^*V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where we define the $f^*(V) = \{(x, v) \in X \times V \mid f(x) = p(v)\}$, and then a fiber bundle on the right induces a fiber bundle on the left, where the fibers are the same on both bundles. Also recall that $V_k \to Gr_k(\mathbb{R}^n)$ is the tautological k-plane bundle defined in the last class, where $V_k = \{(H, v) \mid H \subseteq \mathbb{R}^n, \dim(H) = k, v \in H\}$. Finally recall the definitions of maps of vector bundles. Suppose $V \to X, W \to X$ are vector bundles on X, then a map betwen them is a map $V \to W$ such that the obvious maps commute:

$$\begin{tikzcd} V \arrow[rr, "g"] \arrow[dr] & & W \arrow[dl] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} V \times_X V \arrow[r] \arrow[d, "g\times g"] & V \arrow[d, "g"] \\ W \times_X W \arrow[r] & W \end{tikzcd}$$

and the restriction to each fiber $g|_{V_x} : V_x \to W_x$ is a linear map.

Proposition 20. *Suppose V, W are vector bundles over X, $g : V \to W$ is a map between vector bundles, then the set $\{x \in X \mid g|_{V_x} \text{is an isomorphism}\}$ is open.*

Proof. Suppose $x \in X$ is a point such that $g|_{V_x}$ is iso, then we need a neighborhood U such that the obvious thing holds. Choose a local trivial neighborhood $U' \subseteq X$ of x such that $V|_{U'} = U' \times \mathbb{R}^k$, $W|_{U'} = U' \times \mathbb{R}^k$. Then $g(x, v) = (x, \rho(x)v)$ where $\rho : U' \to M_n(\mathbb{R})$. Now $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ is open, so we can let $U = \{y \in U' \mid \det(\rho(y)) \neq 0\}$. ☐

Now recall the partition of unity: suppose $\{U_\alpha\}$ is an open cover of a paracompact Hausdorff space X. Then there exists a set of functions $\{\rho_\alpha : X \to I\}$ such that $\mathrm{supp}_\alpha \rho_\alpha \subseteq U_\alpha$, $\rho_\alpha(x) \neq 0$ for only finitely many α for each x, and $\sum_\alpha \rho_\alpha(x) = 1$ for all x. [15]

Proposition 21. *Suppose $V \to X$ is a vector bundle, $Z \subseteq X$ is closed, and $s : Z \to V$ is a section, then there exists a section $s' : X \to V$ extending s.*

Proof. First suppose $V = X \times \mathbb{R}^k$, then a section is a map $s(x) = (x, t(x))$ for some $t : X \to \mathbb{R}^k$. Then the case holds by Tietze's theorem. Now let X be covered by local trivial neighborhoods $\{U_\alpha\}$, where $V|_{U_\alpha} = U_\alpha \times \mathbb{R}^k$, and let $Z_\alpha = Z \cap U_\alpha$. Then by case 1, we have some s'_α extending s_α for each s_α defined by restricting s to Z_α. Now, choose a partition of unity ρ, then we claim that $\rho_\alpha s'_\alpha$ define a section on X, and $\sum_\alpha \rho_\alpha s'_\alpha$ is a section extending s on X. ☐

Theorem 27.2. *If V is a vector bundle over $X \times I$, X compact Hausdorff, then $V_0 \sim V_1$, where V_t is the pullback of V along $x \mapsto (x, t)$.*

Proof. Note that we get a function $I \to \mathrm{Vect}_k(X)$ where the right side is the isomorphism classes of vector bundles. We will show that this map is locally constant, and thus constant since I is connected. More precisely, fix some $t \in I$. Compare V and π^*V_t where $\pi : X \times I \to X$ is the projection. Over $X \times \{t\}$, these two things are the same, so they are isomorphic in a neighborhod of $X \times \{t\}$. Since X is compact, there is an isomorphism on $X \times (t - \epsilon, t + \epsilon)$ for some $\epsilon > 0$, which shows that $\forall s \in (t - \epsilon, t + \epsilon)$, $V_s \sim V_t$. ☐

[15] Note that D^n is paracompact Hausdorff, so in fact all CW complexes are paracompact Hausdorff.

Corollary 11. *$Vect_k(X)$ is a homotopy functor of X. In other words, if $f, g : X \to Y$ are homotopic, and $V \to Y$ is a vector bundle, then $f^*V \sim g^*V$.*

Here's a generalization to the theorem above, whose proof we skip:

Proposition 22. *Suppose X is a paracompact Hausdorff, and $V \to X \times I$ is a vector bundle, Write $V_0 \to X$ to be the restriction of V to $X \times \{0\}$. Write $\pi : X \times I \to X$ to be the projection. Then there is an isomorphism $V \cong \pi^* V_0$ extending the identity map over $X \times \{0\}$.*

Note that this is not true for categories in which one doesn't have partitions of unity, e.g. for algebraic vector bundles.

Proposition 23. *Suppose X is compact Hausdorff, $V \to X$ is a vector bundle, then for $n \gg 0$ there is a surjective map*

$$\begin{tikzcd} X \times \mathbb{R}^n \arrow[r, two heads] & V \arrow[d] \\ & X \end{tikzcd}$$

Proof. Let $U_1, \dots, U_m$ be a finite open cover such that $V|_{U_i}$ is trivial. Let $e_i : U_i \to \mathbb{R}^k \to V|_{U_i}$ be the isomorphism, and choose a partition of unity ρ_i, then $\rho_i e_i$ is surjective on the support of ρ_i, so $X \times \bigoplus_i \mathbb{R}^k \xrightarrow{\sum_i \rho_i e_i} V$ is surjective. Note that if X is locally contractible, we can choose $U_1, \dots, U_m$ to work for all vector bundles. $\square$

Finally, let t be the surjective map above, such that $\mathbb{R}^n \xrightarrow{t} V_x$. For each $x \in X$, define $H_x \subseteq \mathbb{R}^n$ to be the orthogonal complement of the kernel of t. Define $f : X \to Gr_k(\mathbb{R}^n)$ that sends x to H_x, then t gives an isomorphism $V \to f^* V_k$.

28 Stiefel-Whitney Classes (Guest Lecturer: Xiaolin Shi)

Recall that we showed in the last lecture that there is a correspondance $\mathrm{Vect}_n(X) \mapsto [X, Gr_n(\mathbb{R}^\infty)]$, and all vector spaces are pullbacks of universal bundles. Today we talk about characteristic classes, which are data that we attach to vector bundles, which can be used to identify whether the bundle is trivial or not. They are to vector bundles as (co)homology are to spaces. (The following construction can be found in [VBKT].)

28.1 Stiefel-Whitney Classes

Definition 27. *Let $V \to X$ be a n-dimensional vector bundle. The* ***Stiefel-Whitney classes*** *$w_i(V)$ are some particular elements of $H^i(X; \mathbb{Z}_2)$.*

There are two ways of constructing S-W classes. One relies on geometry (using BO and BU); a cleaner way is to uniquely identify the S-W classes with some properties, then construct them algebraically from cohomology. We'll go with the second approach.

Proposition 24. *The following properties hold for S-W classes:*

1. *Naturality. In the following pullback of vector bundles:*

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

   The S-W classes $w_i(V) \in H^i(X)$ pulls back to $f^* w_i(v) \in H^i(X')$, and naturality says they are precisely $w_i(V')$.

2. *Cartan formula: Given $V_1 \to X, V_2 \to X$, one can canonically construct the direct sum bundle $V_1 \oplus V_2 \to X$. Define the* ***total S-W class*** *$w(V) = 1 + w_1(V) + w_2(V) + \dots$, then $w(V_1 \oplus V_2) = w(V_1) \cdot w(V_2)$.*

3. *Triviality: $w_i(V) = 0$ for $i > \dim V$.*

4. *Non-triviality: Consider $Gr_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$, there is a* ***(canonical) line bundle*** *$V_1 \to \mathbb{RP}^\infty$, which only has dimension 1 and thus only a single S-W class $w_1(V_1) \in H^1(\mathbb{RP}^\infty)$. The condition is that $w_1(V_1) \neq 0$.*

The Construction Take a vector bundle $V \to X$, and projectivize it to get $P(V) \to X$, where we then have a fiber bundle $\mathbb{RP}^n \to P(V) \to X$. There is a canonical line bundle L over $P(V)$, such that it pulls back to $V_1 \to Gr_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$. Thus we have the following graph:

$$\begin{tikzcd} & L \arrow[r] \arrow[d] & V_1 \arrow[d] \\ \mathbb{RP}^{n-1} \arrow[r] & P(V) \arrow[r] \arrow[d] & Gr_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty \\ & X & \end{tikzcd}$$

Then take the induced map on H^* on the middle line to get $H^*(\mathbb{RP}^\infty) \to H^*(P(V)) \to H^*(\mathbb{RP}^{n-1})$ that sends generators to generators (nontrivial; needs a check). Thus there exists a $x \in H^1(P(V))$ so that the restriction of x to $H^1(\mathbb{RP}^{n-1})$ is a generator. Then by Leray-Hirsch, $H^*(P(V)) = H^*(X)[1, x, x^2, \dots, x^{n-1}]$, which means that x^n must be linearly expressible in $1, x, \dots, x^{n-1}$. Then define w_i to be the coefficients such that $x^n + w_1(v)x^{n-1} + w_2(v)x^{n-2} + \dots + w_n(v) = 0$. To be complete, define higher S-W classes to be trivial.

The Properties Let's check naturality. Take the following pullback:

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

Let x_V and $x_{V'}$ be the canonical generators defined above. We know that $f^*x_V = x_{V'}$, then from the uniqueness of the linear relation above we immediately have the naturality.

Triviality is trivial. Let's now see non-triviality. Take the projectivization $P(V_1) \to \mathbb{RP}^\infty$, what is it? It's really just the identity map $\mathbb{RP}^\infty \to \mathbb{RP}^\infty$. Now let the canonical generator be x, then we have $x + w_1(V_1) \cdot 1 = 0$, thus $x = w_1(V_1) \neq 0$ since we work on $\mathbb{Z}_2$.

Finally, let's check the Cartan formula. Take the projectivization $P(V_1 \oplus V_2) \to X$, and there are two open sets U_1, U_2 covering the total space $P(V_1 \oplus V_2)$, given as $U_i = P(V_1 \oplus V_2) - P(V_i) \cong P(V_{3-i})$, as one can check. Now let $\dim V_1 = m, \dim V_2 = n$. Then define $w_{V_i} \in H^*(P(V_1 \oplus V_2))$, where $w_{V_1} = x^m_{V_1 \oplus V_2} + w_1(V_1)x^{m-1}_{V_1 \oplus V_2} + \ldots + w_m(V_1) \cdot 1 = 0$, and the similar equality holds for w_{V_2}. More concretely, take $U_i \hookrightarrow P(V_1 \oplus V_2)$, and take the cohomology map $H^*(P(V_1 \oplus V_2)) \to H^*(U_i)$, and consider the image w_{V_i}: since $x_{V_1 \oplus V_2}$ pulls back to x_{V_i}, both images w_{V_1} and w_{V_2} would vanish as we pull back, by the defining relations. So the classes are nontrivial, but trivial after being pulled back. In other words, $w_{V_i} \in H^*(P(V_1 \oplus V_2), U_i)$.

Now we claim that $w_{V_1} \cdot w_{V_2} = 0$, which follows the following diagram:

$$\begin{tikzcd}
H^*(P(V_1 \oplus V_2), U_1) \times H^*(P(V_1 \oplus V_2), U_2) \arrow[r] \arrow[d] & H^*(P(V_1 \oplus V_2), U_1 \cup U_2)) = 0 \arrow[d] \\
H^*(P(V_1 \oplus V_2)) \times H^*(P(V_1 \oplus V_2)) \arrow[r] & H^*(P(V_1 \oplus V_2))
\end{tikzcd}$$

Now the Cartan formula comes out by multiplying the coefficients in $0 = w_{V_1 \oplus V_2} = w_{V_1} \cdot w_{V_2}$.

To prove uniqueness we need an important principle:

Proposition 25 (Splitting Principle)**.** *Let $V \to X$ be a dimensional n vector bundle, then there is a pullback*

$$\begin{tikzcd}
V' = L_1 \oplus \ldots L_n \arrow[r] \arrow[d] & V \arrow[d] \\
F(X) \arrow[r] & X
\end{tikzcd}$$

such that V' is the direct sum of n line bundles, and $f^* : H^*(X) \to H^*(F(X))$ is an injection.

Proof. The strategy is to split off a line bundle once at a time. Start with $V \xrightarrow{\pi} X$. Pull it back along $P(V) \xrightarrow{P(\pi)} X$ to obtain a new bundle $P(\pi)^*V \to P(V)$. We know that there's a canonical ling bundle L lying over $P(V)$, so $P(\pi)^*V = L \oplus L^\perp$ (fiberwise orthogonal complement). Do this construction repeatedly and then we're done. Finally, note that the cohomology map on the bottom is $H^*(X) \to H^*(X)[1, x, \ldots, x^{n-1}]$ so it's injective. $\square$

Now suppose $V \to X$ is the canonical line bundle. Then triviality and non-triviality specifies all S-W classes on the canonical line bundle. By naturality, this specifies S-W classes for all line bundles (since they are all pullbacks of the canonical bundle), then by the splitting principle and the Cartan formula, along with the injectivity, this allows us to specify S-W classes for all vector bundles.

29 More on S-W Classes

29.1 Examples of Stiefel-Whitney Classes

Consider the class $\mathbb{RP}^n$, over which there is the totalogical line bundle $L \to \mathbb{RP}^n$. We saw that the total S-W class $w_t(L) = 1 + tx$ (this is a new notation: $w_t(L) = \sum_i w_i(V)t^i$ where t is a formal variable), thus $w_1(L) = x$, where $0 \neq x \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$. Let $V = \bigoplus_k L$, then $w_t(V) = w_t(L)^k = (1+tx)^k = \sum \binom{k}{i} t^i x^i$, then $w_i = \binom{k}{i} x^i$. Now note that L lies in the trivial bundle $\mathbb{R}^{n+1}$, and we have a short exact sequence of vector bundles $0 \to L \to \mathbb{R}^{n+1} \to H \to 0$, where H is called the hyperplane bundle, and the SES is called the **Euler Sequence**. It is a fundamental object in topology and geometry.

In algebraic geometry, this sequence does not split, but in topology the situation is different, as

Proposition 26. *Every short exact sequence of vector bundles split.*

Thus we know that $L \oplus H$ is the trivial bundle, thus $w_t(L) \cdot w_t(H) = 1$, and thus $w_t(H) = \dfrac{1}{1+tx} = \sum t^i x^i$. We know that $\dim H = n$, so $w_{n+1}(H) = 0$, and thus $w_{n+1} = x^{n+1} = 0$. It is also straightforward to show that

Proposition 27. *Every vector bundler over $\mathbb{R}$ is isomorphic to its dual.*

Next, for every $l \in L$, one can consider a neighborhood of l (and thus the tangent space at l) as the graph of linear maps $l \to H$, and thus we see that the tangent bundle $T\mathbb{RP}^n = \mathrm{Hom}(L, H)$. Note that $\mathrm{Hom}(L, \mathbb{R}) = L$. Now we hom into the exact sequence above to get

$$0 \to \mathrm{Hom}(L, L) = * \to \mathrm{Hom}(L, \mathbb{R}^{n+1}) = \oplus_{n+1} L^* = \oplus_{n+1} L \to \mathrm{Hom}(L, H) = T \to 0$$

Thus $w_t(T) \cdot 1 = (1+tx)^{n+1}$, hence the S-W class of $T\mathbb{RP}^n$ is $w_k = \binom{n+1}{k} x^k$.

29.2 Detour: Riemannian metric

Let V be a vector space.

Definition 28. *A Riemannian metric on V is a symmetric bilinear map $V \times V \xrightarrow{\langle \cdot \rangle} \mathbb{R}$ such that it is positive definite: $\langle v, v \rangle > 0$ for $v \neq 0$.*

Definition 29. *A Riemannian metric on a vector bundle is a symmetric bilinear map $V \times_X V \xrightarrow{\langle \cdot \rangle} X \times \mathbb{R}$ that is positive definite.*

Proposition 28. *Every vector bundle over a paracompact space has a Riemannian metric.*

Proof. Obvious when the bundle is trivial. For general $V \to X$, choose a locally trivial covering $\{U_\alpha\}$. Choose a metric $\langle \cdot \rangle_\alpha$ for each local neighborhood, let s_α be a partition of unity, then $\sum_\alpha s_\alpha \langle \cdot \rangle_\alpha$ is a Riemannian metric over V. $\square$

Example 18. *If $0 \to U \to V \to W \to 0$ is an exact sequence of vector bundles, then given a metric $\langle \cdot \rangle$ over V, definte $W' = U^\perp \subseteq V$ w.r.t. the metric, then $W' \to V \to W$ is an isomorpism, so $V = U \oplus W$.*

Question "13" What is the smallest k such that $\mathbb{RP}^{13}$ immerses into $\mathbb{RP}^{13+k}$?

Recall that an immersion is a map such that the induced map of tangent spaces is injective everywhere.

For $f : M \to \mathbb{R}^n$ an immersion, define $\nu \to M$ such that ν_x is the orthogonal complemenet of $DfT_x \subseteq \mathbb{R}^n$. Then we know that

- ν has dimension $n - \dim M$.
- $\nu \oplus T \cong M \times \mathbb{R}^n$.
- $w_t(\nu) \cdot w_t(T) = 1$.

Now, note that $T\mathbb{RP}^{13} \oplus 1 = 14L$, so $w_t(T) = (1+tx)^{14}$, and $w_t(\nu) = (1+tx)^{-14} = \dfrac{(1+tx)^2}{(1+tx)^{16}} = \dfrac{1+t^2x^2}{1+t^{16}x^{16}} = 1+t^2x^2$. Thus $w_1(\nu) = 0, w_2(\nu) \neq 0$. Conclusion: $\mathbb{RP}^{13}$ does not immerse into $\mathbb{RP}^{14}$, for if it did, $w_2(\nu)$ would be 0 since $\dim \nu = 1$. Along the same way, we can conclude that $\mathbb{RP}^9$ is not immersive into $\mathbb{RP}^{14}$, but it might immerse into $\mathbb{RP}^{15}$.

Of course, this does not allow you to construct the immersions. Well, we do have some results:

Theorem 29.1 (Whitney)**.** *Every smooth manifold of dimension n immerses in $\mathbb{R}^{2n-1}$.*

In fact, we have something better:

Theorem 29.2 (P. Cohen)**.** *Every smooth compact manifold of dimension n immerses in $\mathbb{R}^{2n-\alpha(n)}$, where α is the Hamming weight.*

Using K-theory, one can obtain some better results using more sophisticated cohomology theories. In the end, however, the answer to the question "What is the smallest k such that $\mathbb{RP}^n \to \mathbb{R}^{n+k}$" is not known up till this day. The best results we have are obtained by considering this problem in terms of the *axial map problem*, which is to consider the following diagram, where k is the same answer as above:

$$\begin{tikzcd} \mathbb{RP}^n \times \mathbb{RP}^n \arrow[r] \arrow[dr] & \mathbb{RP}^\infty \arrow[d] \\ & \mathbb{RP}^{n+k} \end{tikzcd}$$

and take the generator x for the cohomology of $\mathbb{RP}^\infty$, which pulls back to $(x+y)$ on the left for generators x, y, and try to argue how the graph works when we raise to the $(n+k)$th power.

The best result regarding immersion that we have is the following theorem by Hirsch:

Theorem 29.3 (Hirsch)**.** *A n-dimensional smooth manifold M immerses into $\mathbb{R}^{n+k}$ if and only if there exists $\nu \to M$, with $\dim \nu = k$ and that $\nu \oplus T = M \times \mathbb{R}^{n+k}$.*

The result, known as Smale-Hirsch Theory, applies to greater generality:

Theorem 29.4 (Smale-Hirsch)**.** *Let M be a smmooth m-dimensional manifold, and N a smooth n-dimensional manifold. Then if either $m < n$, or $m = n$ and M open in N, then the space Imm(M, N) of smooth immersions of M into N is weakly homotopy equivalent to Mono(TM, TN), the space of vector bundle monomorphisms from TM to TN.*

This idea leads to a more generalized principle known as the h-principle, which has applications in geometry and PDEs. (c.f. here)

30 Cohomology of the Grassmannian

Let $G = \lim_{n\to\infty} Gr_k(\mathbb{R}^{n+k})$, also written as $Gr_k(\mathbb{R}^\infty)$, which is also $BO(k)$ (the classifying space of $O(k)$). We showed that for paracompact Hausdorff X, $[X, G] \cong \mathrm{Vect}_k(X)$, given by pulling back the universal bundle V_k. Let $w_i = w_i(V_k) \in H^i(Gr_k(\mathbb{R}^\infty))$ be the S-W classes of the universal bundle, then S-W classes of all vector bundles are obtained as pullbacks, i.e. $u_i = w_i(f^*V_k) = f^*w_i$.

Theorem 30.1. *$\mathbb{Z}_2[w_1, \dots, w_n] \to H^*(G)$ is an isomorphism.*

Theorem 30.2. *On $Gr_k(\mathbb{R}^{n+k})$, we have V_k and its orthogonal complement $V_k^\perp$ of dimension n, having S-W classes $w'_1, \dots, w'_n$. Then the map $\mathbb{Z}_2[w_1, \dots, w_k, w'_1, \dots, w'_n]/\{w_i \cdot w'_i = 1 \text{ for all } i\} \to H^*(Gr_k(\mathbb{R}^{n+k}))$ is an isomorphism.*

We shall prove the first theorem only.

Preparation: Cellular Structure of Grassmannians Recall the cell structure on $Gr_k(\mathbb{R}^{n+k})$: there is a cell corresponding to each sequence $0 \le a_1 \le \dots \le a_k \le n$, with dimension $a_1 + \dots + a_k$. The interior is the k-plane whose basis has form

$$\begin{pmatrix} * & \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \dots & 0 & * & \dots & 1 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & 0 & * & \dots & 0 & * & \dots & 1 & 0 \end{pmatrix}$$

Given some k-plane V in $\mathbb{R}^{n+k}$, associate with it the sequence $\dim(V\cap\mathbb{R}^1), \dim(V\cap\mathbb{R}^2), \dots, \dim(V\cap\mathbb{R}^{n+k})$ where $\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \dots \subseteq \mathbb{R}^{n+k}$ is some chosen sequence of subspaces. This sequence eventually goes to k, but we need to keep track where it bumps up (we say that it bumps up at the "jump" dimensions j_i). After scaling we can put each basis vector as ending in 1 and thus having $j_i - 1$ free dimensions; by subtracting previous basis vectors we can get the form as above. For instance, in the following element of $Gr_3(\mathbb{R}^5)$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

we have the sequence $(\dim(V \cap \mathbb{R}^i)) = (0, 1, 2, 2, 3)$, thus the jump dimensions are $(2, 3, 5)$. By counting the number of asterisks in the matrix above, it is clear that for each jump dimension sequence $0 < x_1 < \dots < x_k \le n+k$, the dimension of the associated cell is $(x_1 - 1) + \dots + (x_k - k)$; taking $a_i = x_i - i$ we get the result above.

Proof of the First Theorem. First let us show that the map is a monomorphism. Consider the classifying map $(\mathbb{RP}^\infty)^n \xrightarrow{L_1\oplus\dots\oplus L_n} Gr_n(\mathbb{R}^\infty)$ of nth product of the tautological line bundle[16] One can check the composition $\mathbb{Z}_2[w_1, \dots, w_n] \to H^*(G) \to H^*((\mathbb{RP}^\infty)^n) = \mathbb{Z}_2[z_1, \dots, z_n]$ sends $w_i \to \sigma_i$, the ith elementary symmetric function in $z_1, \dots, z_n$. We know from classical invariant theory that $\mathbb{Z}_2[z_1, \dots, z_n]^{\Sigma_n} = \mathbb{Z}_2[\sigma_1, \dots, \sigma_n]$; in particular, $\sigma_1, \dots, \sigma_n$ are algebraically independent, so the composite map is injective.

Now we need to show the map is an epimorphism. It suffices to show that for each m, $\dim \mathbb{Z}_2[w_1, \dots, w_k]_m \ge \dim H^m(G)$, since we already showed that it's monomorphic. Consider the **Poincare Series** of the Grassmannian $P_G = \sum_m \dim H^m(G)t^m$, and that of the polynomial ring $P_W = \sum_m \dim \mathbb{Z}_2[w_1, \dots, w_k]_m t^m$.

About Poincare Series Let A_* be a graded ring over $\mathbb{Z}_2$, then $P(A_*) = \sum \dim A_m t^m$. Some properties:

1. $P(A \otimes B) = P(A)P(B)$. (Easy to check.)
2. Suppose $A = \mathbb{Z}_2[x]$ for $|x| = l$, then $P(A) = \dfrac{1}{1-t^l}$.

Thus we know that $P_W = \prod_{i=1}^k \dfrac{1}{1-t^i}$ (this is also known as the partition generating function) $= \sum_i \rho_i t^i$, so ρ_i is the number of ways to write i in the form $s_1 + 2s_2 + \dots + ks_k$. This in turn one-to-one corresponds to a sequence $0 \le a_1 \le \dots \le a_k \le i$, where $a_i = s_1 + \dots + s_i$; this corresponds, as we discussed above, to a cell in G of dimension $\sum_i a_i = i$, so ρ_i equals the number of i-cells in G. Finally, recall that the number of i-cells of G upper bounds the dimension of $H^i(G)$, so we're done. ∎

[16] Recall that given any principal bundle $\pi : EG \to BG$ and any G-principal bundle $\gamma : Y \to X$, there is a classifying map $X \to BG$ such that γ is the pullback of π along this map.

A few words on the second theorem. If we write $w_i \cdot w_i' = 1 + \sum_i r_i$, then $w_i \cdot w_i' = 1$ is equivalent to $r_i = 0\ \forall i$. In fact, the sequence (r_i) form a regular sequence (as defined in commutative algebra).

For instance, consider $Gr_2(\mathbb{R}^5)$, then the cohomology is given by $\mathbb{Z}_2[w_1, w_2, w_1', w_2', w_3']/\{r_1, r_2, \ldots, r_5\}$. Then we have

$$\frac{(1-t)(1-t^2)\ldots(1-t^5)}{(1-t)(1-t^2)(1-t)(1-t^2)(1-t^3)}$$

this is the *Gaussian* binomial coefficients, and they are always polynomials, and one can check that they count the number of cells of Grassmannians of finite dimensions. (Write $\binom{m}{r}_q = \frac{(1-q^m)\ldots(1-q^{m-r+1})}{(1-q)\ldots(1-q^r)}$ if $r \leq m$, and 0 otherwise. Then the expression above is simply $\binom{5}{2}_t$. The general statement is that $\binom{n}{k}_q$ counts k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$. Also, I guess I should mention that this is also used in the study of quantum groups – Charles.)

31 Poincare-Thom Construction

Here we'll use some facts about transversality. Let W, M be manifolds, $Z \subseteq W$ be a submanifold of W. We say a smooth map $f : M \to W$ is transverse in Z if for all $z \in Z$ and $x \in M$ such that $f(x) = z$, we have $T_xM \oplus T_zZ \to T_{f(x)}W$ surjective.

Proposition 29. *Every $M \to W$ is homotopy to one that is transverse in Z. If f is already transverse on a closed $K \subseteq M$, then can choose the homotopy to be constant on a neighborhood of K. Also, there is an extended version for M with boundary.*

Suppose we have a map $f : S^{n+k} \to S^n$, and pick a point $x \in S^n$. We can move f by a homotopy if necessary to make it transverse in x. Suppose $f, g : S^{n+k} \to S^n$, then we can choose a homotopy $H : S^{n+k} \times I \to S^n$ such that it's transverse to x on the two ends of the homotopy. Then there is a homotopy of H to a transverse map, which is constant in a neighborhood. Thus we get a new homotopy H' of f, g that is transverse to x. Let $N = (H')^{-1}(x)$, then N is a cobordism of $f^{-1}(x)$ and $g^{-1}(x)$.

The Goal We would like to have a tool to link homotopy and cobordism. This is given by the Pontryagin-Thom construction, as we describe below.

Suppose $V \to X$ is a vector bundle of dimension n, where X is a smooth manifold. Define $\mathrm{Thom}(X, V) = X^V$, where we one-point compactify every fiber of V, then mod out X. If we give V a Riemannian metric, then we can say it is $B(V)/S(V)$, where $B(V) = \{v \in V \mid \|v\| \leq 1\}$, $S(V) = \{v \in V \mid \|v\| = 1\}$. In what follows, we shall also use $\mathbb{R}$ to denote the trivial bundle $X \times \mathbb{R} \to X$. The Thom space has the following easy to check properties:

- $(X \times Y)^{V \oplus W} = X^V \wedge Y^W$;
- In particular, $X^{V \oplus \mathbb{R}} = \Sigma X^V$.

Now let $f : S^{n+k} \to X^V$, and note that X^V is no longer a smooth manifold, but it is smooth away from the point of infinity. So to handle this we extend the transversality theorem, noting that it is local property:

Remark 12. *In the transversality theorem above, W doesn't have to be smooth in a neighborhood of Z.*

Of course, $X \subseteq X^V$, so let's move f by a homotopy if necessary, then we can assume that f is transverse to X. Let $M = f^{-1}(X) \subseteq S^{n+k}$, then it has dimension k. Then we can interpret the homotopy of X^V in terms of M. To do so, we need some properties about M. It of course comes in with an embedding in S^{n+k}, and also a map $f : M \to X$. Moreover, it also comes from an isomorphism $\nu \sim f^*V$, where ν is the normal bundle to the embedding $M \to S^{n+k}$, given by Df.

Now suppose we have a homotopy $H : S^{n+k} \times I \to X^V$ between $f = H(-, 0)$ and $g = H(0, -)$ that is transverse in X. If $f^{-1}(X) = M_0$, $g^{-1}(X) = M_1$, then $N = H^{-1}(X)$ is a cobordism of M_0 and M_1, which comes with all the associated data that is compatible with the data from the two submanifolds. This gives a well-defined map from $\pi_{n+k}X^V$ to the set $\{$The three maps mentioned above$\}$.

Theorem 31.1 (Pontryagin-Thom)**.** *This map is a bijection.*

Note that:

1. We can let the embedding be $M \to \mathbb{R}^{n+k}$ instead, because clearly the image and thus the preimage are away from the infinity points respectively.
2. What's the additive structure on the RHS set? Move things by a translation we can take the disjoint union of the embeddings, which doesn't change the elements up to cobordism.

Proof. What we describe below is known as the Pontryagin-Thom construction. Suppose we have $M \subseteq \mathbb{R}^{n+k} \subseteq S^{n+k}$, $f : M \to X$ and $\nu \to f^*V$, where $V \to X$ is a vector bundle over a manifold. Recall the tubular neighborhood theorem from baby topology: suppose a manifold M embeds in $\mathbb{R}^{n+k}$, then there exists a neighborhood U of M in $\mathbb{R}^{n+k}$, and a diffeomorphism $U \to \nu$, such that the zero section $M \to \nu$ is the same as the inclusion $M \to U$ posted composed by the diffeomorphism. Thus we can choose a tubular neighborhood U of M in $\mathbb{R}^{n+k}$, and get a continuous map $S^{n+k} \to S^{n+k}/(S^{n+k} - U) \to M^\nu$ such that x maps to x if $x \in U$, and $*$ if $x \notin U$. This map is called the Pontryagin-Thom collapse. But recall that the following diagram:

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & V \arrow[d] \\ M \arrow[r, "f"] & X \end{tikzcd}$$

induces a map $M^\nu \to X^V$, thus the concatenation of this map is the inverse direction that we need. $\square$

A general reference for cobordism is Robert E. Stong's *Notes on Cobordism Theory*, though the book itself might be somewhat hard to read.

32 Pontryagin-Thom and the MO Spectrum

Last class we proved, using transversality, that if $V \to X$ is a vector bundle of dimension n, X is a smooth manifold, then we have the following isomorphism regarding the Thom spectrum (defined in the next lecture – Charles):

$$\pi_{n+k}X^V = \{M \to X, M \to \mathbb{R}^{n+k}, f^*V \sim v\}/\text{cobordism}$$

Remark 13. *X doesn't have to be a smooth manifold.*

Let's observe that it still makes sense to talk about transversality of $S^{n+k} \to X^V$ over some $x \in X^V$, even if X is not a smooth manifold. Consider the map $X \xrightarrow{g} Gr_n(\mathbb{R}^N)$ given in the universal bundle construction, explicitly by $V \mapsto g^*V_n$. Since we are only interested in homotopy group, we can change space up to a homotopy, so might as well suppose that g is a Serre fibration. Then so is

$$\begin{array}{ccc} V & \longrightarrow & V_k \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{g} & Gr_n(\mathbb{R}^N) \end{array}$$

We observe that $X^V \to Gr_n(\mathbb{R}^N)^{V_n}$ is a Serre fibration away from $*$.

Definition 30. *Let L be a manifold with boundary, $L \to X^V$ is said to be transverse to the zero section if $L \to X^V \xrightarrow{g} Gr_n(\mathbb{R}^N)^{V_n}$ is transverse to $Gr_n(\mathbb{R}^N) \subseteq Gr_n(\mathbb{R}^N)^{V_n}$.*

When $X \to Gr_n(\mathbb{R}^N)$ is a Serre fibration, every map $L \to X^V$ is homotopic to a map that is transverse to the zero section. (My raw note says here "we can move a bit such that we don't move in a neighborhood of the base point." I'm not sure what's going on here. – Charles)

Now suppose we have the data $(M \xrightarrow{f} X, M \xrightarrow{\iota} \mathbb{R}^{n+k}, f^*V \sim v)$, then we get an element of the homotopy group. Suppose $\iota_1, \iota_2 : M \to \mathbb{R}^{n+k}$ are two isotopic embeddings, i.e. there is an embedding $M \times I \xrightarrow{h} \mathbb{R}^{n+k} \times I$ that restricts to ι_i on two ends, then this h gives a cobordism between $(M, f^*V \sim v_1, \iota_1)$ and $(M, f^*V \sim v, \iota_2)$, so they represent the same homotopy class. Thus the homotopy element depends only on the isotopy type of ι. However, for $N \gg 0$, by a general position argument we see that any two embeddings $M \to \mathbb{R}^n$ are isotopic, (so the space of embeddings $\lim_{n\to\infty} \text{Emb}(M, \mathbb{R}^n)$ is contractible, because we can take M to be $S^n \times M$), so if we can embed into even larger spaces we can probably get rid of ι. Well, if we admit to a larger embedding $M \to \mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$, then the normal bundle data becomes $f^*V \oplus \mathbb{R} \sim v \oplus \mathbb{R}$. Then the new data corresponds to the homotopy $S^{n+k+1} \to X^{V\oplus\mathbb{R}} = \Sigma X^V$. If $r : S^{n+k} \to X^V$, then this new map is simply Σr. Thus if we consider $\lim_{t\to\infty} \pi_{n+k+t}\Sigma^t X^V$, then this can be specified by the data $M \xrightarrow{f} X$ and $f^*V =_s \nu$, where $=_s$ is a stable isomorphism (see below).

At this moment it is fitting to introduce some more notation. Let us write $\pi_n^s Z = \lim_{t\to\infty} \pi_{n+t}\Sigma^t Z$ and call it the nth stable homotopy group. For instance, by Freudenthal suspension theorem, stable homotopy groups $\pi_k^s S^0$ are well-defined.

Definition 31. *Two vector bundles V, W on X are stably isomorphic if $V \oplus \mathbb{R}^N \sim W \oplus \mathbb{R}^N$ for $N \gg 0$.*

Example 19. *$TS^n \to S^n$ and $S^n \times \mathbb{R}^n \to S^n$ are not isomorphic unless $n = 1, 3, 7$, in which cases the sphere has trivial tangent bundle. However, these two bundles are stably isomorphic, for $TS^n \oplus \mathbb{R} \cong S^n \times \mathbb{R}^n \times \mathbb{R}$.*

Remark 14. *In terms of commutative algebra, any vector bundle is a finitely generated projective module over the space of continuous functions over the base space; then $S^n \times \mathbb{R}^n \to S^n$ is a free module, so we have an example of a projective module that, when summed with a free module, becomes free.*

Now we'd like to remove the normal bundle data as well. Is there a universal X such that we don't have to mention that data? Well, suppose M is of dimension k, then the stable isomorphism $f^*V =_s \nu$ is given by that $TM \oplus f^*V \oplus \mathbb{R}^j \cong \mathbb{R}^{n+k+j}$. Note that M embeds into f^*V, so there is a mapping $M \to Gr_k(\mathbb{R}^{n+k+j})$. Thus the tangent bundle / normal bundle is classified by a map into the Grassmannian. Thus if we take $X = Gr_{n+j}(\mathbb{R}^{n+j+k})$ and $V = V_{n+j}$, the data $M \to X$ already gives the normal bundle data as well.

The structure that we eventually arrive at is the sequence of spaces $Gr_n(\mathbb{R}^\infty) = BO(n)$. It has a universal n dimensional bundle V_n, and the Thom complex $\text{Thom}(Gr_n, V_n) = MO(n)$. Note that there is a map $Gr_n(\mathbb{R}^\infty) \to Gr_{n+1}(\mathbb{R}^\infty \oplus \mathbb{R}) = Gr_{n+1}(\mathbb{R}^\infty)$, and the V_{n+1} pulls back to $V_n \oplus \mathbb{R}$, then we see that $\pi_{n+k}MO(n) \to \pi_{n+k+1}\Sigma MO(n) \to \pi_{n+k+1}MO(n+1)$, and one observes that $\lim_{n\to\infty} \pi_{n+k}MO(n) = $ k-manifolds/cobordism. Here we are looking at the homotopy group of a spectrum.

33 Spectra

We have proven that the set of cobordism classes of k manifolds is the same as $\lim_{n\to\infty} \pi_{n+k}MO(n)$ where $MO(n)$ is the Thom complex $\text{Thom}(Gr_k(\mathbb{R}^\infty))$. Note that we have the following diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & Gr_k(\mathbb{R}^\infty) \arrow[d] \\ \mathbb{R}^{n+k} \arrow[r] & f^*V_k \end{tikzcd}$$

We should be able to multiply with another space X, and add the trivial bundle for X, then because $(X \times Y)^{V\oplus W} = X^V \wedge Y^W$, we also obtain $\pi_{n+k}(MO(n) \wedge X)$. So we also observe that the cobordism classes of k-manifolds $M \times X$ is $\lim_{n\to\infty} \pi_{n+k}(MO(n) \wedge X)$.

Definition 32 (Spectrum)**.** *A spectrum E consists of a sequence of spaces E_n and connecting maps $t_n : \Sigma E_n \to E_{n+1}$.*

Example 20. *$E = \{S^n\}$, where $E_n = S^n$ and $\Sigma S^n \to S^{n+1}$ is the usual map. This is the sphere spectrum S^0.*

In general, if X is any pointed space, there is a suspension spectrum $\Sigma^\infty X$ given by just $E_nX = \Sigma^n X$.

Example 21. *The spectrum $MO = \{MO(n)\}$.*

Example 22. *Suppose E is a spectrum and X a pointed space, then we can form $E \wedge X = \{E_n \wedge X\}$. For instance, $\Sigma^\infty X = S^0 \wedge X$.*

Definition 33. *E is an Ω-spectrum if each of the maps $E_n \to \Omega E_{n+1}$, which are the adjoints of t_n, is a weak equivalence.*

In some discussions, what we call "spectrum" is called a "prespectrum," and "Ω-spectra" are called "spectra". (This is used primarily in homological algebra.) Now if E is a spectrum, we set $\pi_k E = \lim_{n\to\infty} \pi_{n+k}E_n$. There is a lot of details in the construction of spectra, but we'll not go deep into the theory here. Note that of course the definition $\pi_k E$ makes sense for $k \in \mathbb{Z}$ since eventually $(n + k)$ is positive. For example, consider $m > 0$ and let S^{-m} be the spectrum given by $S^{-m}_j = *$ for $j < m$, and $S^{-m}_j = S^{j-m}$ for $j \geq m$. Then $\pi_{-m}S^{-m} = \lim_{n\to\infty} \pi_{n-m}S^{-m}_n = \lim_{n\to\infty} \pi_{n-m}S^{n-m} = \mathbb{Z}$.

Definition 34. *A map $E \to F$ of spectra consists of a collection of maps $E_n \to F_n$ such that it commutes with t_n. It is a weak equivalence if it induces an isomorphism of π_k for all $k \in \mathbb{Z}$.*

Proposition 30. *Every spectrum is weakly equivalent to a Ω-spectrum.*

Proof. Given $E = \{E_n\}$, define a new spectrum $\overline{E}$ given by $\overline{E}_n = \lim_{m\to\infty} \Omega^m E_{n+m}$. (We ignore some technical details.) Then there is a canonical mapping $E \to \overline{E}$, the latter of which is an Ω-spectrum. $\square$

Proposition 31. *If E is an Ω-spectrum, then for $n + k \geq 0$, the map $\pi_{n+k}E_n \to \pi_{n+k+1}E_{n+1}$ is an isomorphism.*

Proposition 32. *Suppose E is a spectrum, X is a space. Then the obvious map $\pi_k(E \wedge X) \to \pi_{k+1}(E \wedge \Sigma X)$ is an isomorphism.*

The proof below is sometimes called a *roller-coaster* proof, due to its associated diagram.

Proof Sketch. We have the following diagram:

$$\begin{tikzcd} \pi_{n+k}(E_n \wedge X) \arrow[r] \arrow[d] & \pi_{n+k+1}(E_{n+1} \wedge X) \arrow[r] \arrow[d] & \pi_{n+k+2}(E_{n+2} \wedge X) \arrow[r] \arrow[d] & \dots \\ \pi_{n+k+1}(E_n \wedge \Sigma X) \arrow[r] \arrow[ur] & \pi_{n+k+2}(E_{n+1} \wedge \Sigma X) \arrow[r] \arrow[ur] & \pi_{n+k+3}(E_{n+2} \wedge \Sigma X) \arrow[r] \arrow[ur] & \dots \end{tikzcd}$$

Now note that the colimits of the two rows must be the same. (The actual proof is not this easy and involves some careful detail checking.) $\square$

Proposition 33. *Suppose $A \to X$ is a map, E is a spectrum. Then there is a LES*

$$\ldots \to \pi_k(E \wedge A) \to \pi_k(E \wedge X) \to \pi_k(E \wedge (X \cup CA)) \xrightarrow{\delta} \pi_{k-1}(E \wedge A) \to \ldots$$

where δ is given by $\pi_k(E \wedge (X \cup CA)) \to \pi_k(E \wedge \Sigma A) \xleftarrow{\cong} \pi_{k-1}(E \wedge A)$.

Proof. We just need to check exactness at $\pi_k(E_n \wedge X)$ since, for instance, exactness at $\pi_k(E \wedge (X \cup CA))$ is exactness of the middle term in the LES associated with $X \to (X \cup CA) \to (X \cup CA) \cup CX = \Sigma A$. Note that $\pi_k(E \wedge A) \to \pi_k(E \wedge X) \to \pi_k(E \wedge (X \cup CA))$ is zero because $A \to X \to X \cup CA$ is zero. We have the following diagram:

$$\begin{tikzcd} S^{n+k} \arrow[r] \arrow[d, "x"] & D^{n+k+1} \arrow[r] \arrow[d] & S^{n+k+1} \arrow[d, "y"] \\ E_n \wedge X \arrow[r] & E_n \wedge (X \cup CA) \arrow[r] & E_n \wedge \Sigma A \end{tikzcd}$$

The y in the diagram is defined by the diagram and is an element of $\pi_{k+1} E \wedge \Sigma A = \pi_k E \wedge A$. And it turns out that y gets sent to x in the first map. To see this, extend the map to the right again to have the bottom be $E_n \wedge \Sigma X$, then check that the vertical arrow is again x. ☐

Corollary 12. *A spectrum gives a generalized homology theory $(X, A) \mapsto E_k(X, A) = \pi_k E \wedge (X \cup CA)$.*

Example 23 (The singular homology spectrum)**.** *Let $HA = \{K(A, n)\}$, where $\Sigma K(A, n) \to K(A, n+1)$ is given by the adjoint of $K(A, n) \xrightarrow{\cong} \Omega K(A, n+1)$. Then $HA_k(X) = \lim_{n\to\infty} \pi_{n+k} K(A, n) \wedge X$. When $X = *$, $\pi_k(HA) = 0$ if $k \neq 0$ and A if $k = 0$, so $HA_k(\bullet)$ also satisfies the dimension axiom. So in fact $HA_k(X) = H_k(X; A)$. In other words, $H_k X = \lim_{n\to\infty} \pi_{n+k}(K(A, n) \wedge X)$.*

Theorem 33.1. *Every homology theory comes from a spectrum.*

Definition 35 (Cohomology Theory of Spectra)**.** *Define $E^k(X, A) = [\Sigma^n(X \cup CA), \overline{E}_{n+k}]$ for some large n.*

34 Thom Isomorphism

There is a relative version of the Serre spectral sequence that goes as follows: given a Serre spectral sequence $F \to E \xrightarrow{p} B$, and subspaces $F_0 \subseteq F$, $E_0 \subseteq E$ such that $F_0 \to E_0 \xrightarrow{p} B$ is again a Serre fibration, then for each subspace $B_0 \subseteq B$, we have a spectral sequence $H^p(B, B_0; H^q(F, F_0)) \Rightarrow H^{p+q}(E, E_0 \cup p^{-0}(B_0))$. In particular, taking B_0 to be a point, we get $H^p(B; H^q(F, F_0)) \Rightarrow H^{p+q}(E, E_0)$. Note that if $\pi_1 B$ does not act trivially, we'll need to use local coefficients.

Now consider a vector bundle V of dimension n over a space X. Then the pointwise fiber is $\mathbb{R}^n$. Consider the sphere bundle S^V whose pointwise fiber is the one-point compactification, i.e. S^n. Then as we defined before, $\mathrm{Thom}(X, V) = S^V/X$. Now the trivial bundle $* \to X \to X$ is a subbundle of $S^n \to S^V \to X$ in the sense above, so applying the spectral sequence we obtain $H^p(X; H^q(S^n, *)) \Rightarrow H^{p+q}(\mathrm{Thom}(X, V))$ (we use local coefficients if applicable).

Corollary 13. *$H^{n+p}(\mathit{Thom}(X, V)) \cong H^p(X; H^n(S^n, *))$.*

Example 24 (An example of Thom isomorphism)**.** *Consider mod 2 cohomology. We have $H^n(S^n) = \mathbb{Z}_2$, and $\mathit{Aut}(\mathbb{Z}_2)$ is trivial, so $\pi_1 X$ acts trivially, and thus $H^{*+n}(\mathit{Thom}(X, V)) \cong H^*(X)$.*

Thom Isomorphism Suppose V is a vector bundle over X of dimension n. A *Thom class* for V is an element $u \in H^n(\mathrm{Thom}(X, V))$ having the property that, for each $x \in X$, the image of u in $H^n(S^V_x)$ (where S^V_x is the fiber over x of $S^V \to X$) is a generator.

Observation: $\widetilde{H}^*(\mathrm{Thom}(X, V))$ is a module over $H^*(X)$. To see this, observe that $H^*(\mathrm{Thom}(X, V)) = H^*(S^V, X)$, and the action is given by the g map in the following diagram:

$$\begin{tikzcd}
H^*(S^V) \otimes H^*(S^V, X) \arrow[r, "\cup"] & H^*(S^V, X) \\
H^*(X) \otimes H^*(S^V, X) \arrow[u, "p^* \otimes 1"] \arrow[ur, "g"] &
\end{tikzcd}$$

where p^* is the map induced by $p : S^V \to X$, and $\cup$ is the relative cup product. Note that the presentation of $\mathrm{Thom}(X, V)$ is different from the standard one, where it is considered as $B(V)/S(V)$. Another way to see the module structure is by considering $H^*(\mathrm{Thom}(X, V)) = H^*(V, V - X)$.

Theorem 34.1 (Thom isomorphism)**.** *Multiplication by a Thom class u (i.e. $b \mapsto g(b, u)$) is an isomorphism from $H^*(X) \to \widetilde{H}^{*+n}(\mathit{Thom}(X, V))$.*

It is possible to prove this using Serre spectral sequences. Here's another proof sketch:

1. First prove it for $V = X \times \mathbb{R}^n$, the trivial bundle, using suspension isomorphism;
2. Then by Meyer-Vietoris, we can see it holds when X is covered by finitely many sets on which V is trivial;
3. Pass to the limit.

Some words on Thom classes. A unique Thom class exists for every V in mod 2 cohomology, since $\tilde{H}^*(\mathrm{Thom}(X, V)) \cong \tilde{H}^*(S^V_x)$ for each x. When we use $\mathbb{Z}$ coefficients it is no longer unique; it is, however, unique for simply connected space. For orientable manifolds, the choice of a Thom class is equivalent to the choice of an orientation.

Let's see some Thom isomorphisms in action.

- Suppose we have M of dimension n, N of dimension $n + d$ being smooth manifolds, and $f : M \hookrightarrow N$, so d is dimension of the normal bundle ν. There is the Pontryagin-Thom collapse map $N \to \mathrm{Thom}(M, \nu)$, which induces $H^{k+d}(\mathrm{Thom}(M, \nu)) \to H^{k+d}(N)$. If we precompose this with the Thom isomorphism $H^k(M) \to H^{k+d}(\mathrm{Thom}(M, \nu))$, we obtain a mapping $H^k(M) \to H^{k+d}(N)$, which at a first glance is "going the wrong way." This map is called the *pushforward map.*
- Same setting as above. Consider the embedding $M \hookrightarrow \mathbb{R}^m \times N \to \mathrm{Thom}(N, \mathbb{R}^m)$ (where $\mathbb{R}^m \times N$ is the trivial bundle of rank m), then Pontryagin-Thom gives a map $\mathrm{Thom}(N, \mathbb{R}^m) \to M^\nu$, where ν is of rank $m + d$. Then we have the following diagram:

$$\begin{tikzcd}
H^{k+d+m}(\mathrm{Thom}(N, \mathbb{R}^m)) \arrow[d] & \widetilde{H}^{k+d+m}(M^\nu) \arrow[l] \\
H^{k+d}(N) & H^k(M) \arrow[l] \arrow[u]
\end{tikzcd}$$

Note that we can obtain this even when $f : M \to N$ is not an embedding. This map has a lot of interpretations, but is often referred to as "integration over the fibers."

We can get this map from another perspective:

$$\begin{array}{ccc} H^k(M) & \longrightarrow & H^{k+d}(N) \\ \downarrow & & \uparrow \\ H_{n-k}(M) & \longrightarrow & H_{n-k}(N) \end{array}$$

where the two vertical maps are given by Poincare duality. In fact, Poincare duality is just Thom isomorphism with some additional constructions.

Take $MO(k) = \text{Thom}(Gr_k(\mathbb{R}^\infty))$. The kth cohomology of this object is $\mathbb{Z}_2 = H^0(Gr_k(\mathbb{R}^\infty))$. Now consider $MO(k) \xrightarrow{u} K(\mathbb{Z}_2, k)$; as we'll see, it induces monomorphism on cohomology through a certain range. The question is, what would be the Steenrod operations of u? i.e. what is $Sq^j(u)$?

Proposition 34. *$Sq^j(u) = g(w_j, u)$, which we'll just write as $w_j u$ from now on.*

Proof Sketch. By Thom isomorphism, $Sq^j(u) = w'_j u$ for some w'_j which are the S-W classes of *some bundle* V over X. One then check that they satisfy 1) naturality: for every $f : X \to Y$, we have $w'_j(f^*V) = f^*(w'_j(V))$; 2. the Cartan formula; and 3. $w'_j(L) \neq 0$ where L is the tautological line bundle over $\mathbb{R}^\infty$. But these axioms characterize S-W classes, so $w'_j = w_j$. $\square$

35 Cohomology of MO and Euler Classes

The big theorem today is the following:

Theorem 35.1. *$MO(k) \xrightarrow{u} K(\mathbb{Z}_2, k)$ induces a monomorphism on cohomology through dimension $2k$.*

We know the cohomology of the space on the right, and we know what individual Steenrod operations do on the Thom class: $Sq^i u \mapsto w_i u$, but it is harder to compute general $Sq^I u$ because we have to analyze the S-W classes.

Well, let's look at the proof. For this kind of statement, here's a tried-and-true technique: find something that maps into the first space, and prove that the composition mapping is a monomorphism. (We've seen this a few times before in this note. – Charles)

Definition 36 (Euler Class)**.** *Suppose $V \to X$ is a vector bundle of dimension n. Consider the Thom complex $Thom(X, V)$, and there is a zero section $X \to Thom(X, V)$. Then $u \in H^n(Thom)$ pulls back to the Euler class $e(V) \in H^n(X, \mathbb{Z}_2)$.*

Why is it called the *Euler* class? After all, granted that Euler was a genius, his time was a bit too early for things such as vector bundles and cohomologies. Well, suppose M is a manifold, then $e(TM) \in H^n$ and $\langle e(TM), [*]\rangle = \chi(M)$ by Poincare duality, where χ is the Euler characteristic of manifolds.

Now is a good time to list some properties of Euler class:

Remark 15. *In the next few propositions, tensors, cup product and direct sums are external. In particular, the* external *cup product is what Hatcher calls the cross product of cohomology.*

Proposition 35. *$e(V \oplus W) = e(V)e(W)$.*

Proof. Recall that $\text{Thom}(X \times Y, V \oplus W) = \text{Thom}(X, V) \wedge \text{Thom}(Y, W)$. Thus $u_{V\oplus W} = u_V \otimes u_W$. Then the rest follows from naturality, by looking at the following diagram:

$$\begin{tikzcd}
\text{Thom}(V \otimes W) \arrow[r] & \text{Thom}(V \otimes W) \arrow[r, "\cong"] & \text{Thom}(V) \wedge \text{Thom}(W) \\
X \arrow[u] \arrow[r, "\Delta"] & X \times X \arrow[u] \arrow[ur] &
\end{tikzcd}$$

□

Proposition 36. *If V is the trivial bundle, $e(V) = 0$.*

Proof. The zero section from X to the Thom space of the trivial bundle is null. □

Corollary 14. *If $e(V) \neq 0$, then $V \neq V' \oplus \mathbb{R}$ for any V'.*

Proposition 37. *$e(V) = w_n(V)$.*

Proof. Under the Thom isomorphism, $e(V) = u \cdot e(V) = u \cdot u = Sq^n(u) = w_n(V)$. □

Corollary 15. *If L is a line bundle, then $e(L) = w_1(L)$.*

Remark 16. *Give V a metric, let $S(V)$ be the unit sphere, then $Thom(X, V)$ is the mapping cone of $S(V) \to X$. In particular, $S(V) \to X \xrightarrow{\text{zero section}} Thom(X, V)$ is exact.*

Example 25. *Let L be the tautological bundle over $\mathbb{RP}^\infty$, then $S(L) = S^\infty = *$. Thus in $S(V) \to \mathbb{RP}^\infty \xrightarrow{f} Thom(\mathbb{RP}^\infty, L)$, the map f is a homotopy equivalence, so it induces isomorphism on all cohomologies, in particular u pulls back to a nontrivial class.*

Example 26. *Consider $(\mathbb{RP}^\infty)^k \xrightarrow{L_1 \oplus \ldots \oplus L_k} Gr_k(\mathbb{R}^\infty)$. The second term has Thom complex $MO(k)$, and by the example above, the first term has Thom complex $(\mathbb{RP}^\infty)^{\wedge k}$.*

Now let's go back to $MO(k) \to K(\mathbb{Z}_2, k)$. Consider the composite map $(\mathbb{RP}^\infty)^k \xrightarrow{\text{via } BO(k)} MO(k) \to K(\mathbb{Z}_2, k)$, then ι pulls back to u and then to $e(L_1 \oplus \ldots \oplus L_k) = x_1 \ldots x_k$. Recall the proof of stable cohomology operations being determined by its product-of-one-dimensional behaviors, and by that same argument this induces a monomorphism through a range of dimensions.

At this moment let's revisit something left out a few lectures ago. Recall that $\pi_n MO = \lim_{k\to\infty} \pi_{n+k} MO(k)$. How do we know this actually stabilizes? Well, we have the suspension homomorphism $\Sigma MO(k) \to MO(k+1)$. Recall, on the other hand, that $BO(k) \to BK(k+1)$ is an iso in cohomology through dimension $2k$. Then as we shift to MO via multiplication by Thom class, by what we just showed above, we know that $\Sigma MO(k) \to MO(k+1)$ is iso in cohomology through $2k+1$. Thus $\pi_{n+k}MO(k)$ is independent of k for $k > n$ via a standard Hurewicz argument.

Now a spectra perspective. Recall that we have $u_k : MO(k) \to K(\mathbb{Z}_2, k)$, and it is compatible with suspension of the MO, and thus we have a mapping of spectra $MO \to H\mathbb{Z}_2$.

Definition 37 (Cohomology of Spectra)**.** *If $E = \{E_n\}$ is a spectrum, then $H^j(E) = \lim_{n\to\infty} H^{n+j}(E_n)$.*

Corollary 16. *The map $MO \to H\mathbb{Z}_2$ is a monomorphism on cohomology.*

In general, cohomology of a space is a ring along with Steenrod operations, such that $Sq^n(x) = 0$ for $n > \dim(X)$. On the contrary, cohomology of a spectrum is not a ring (note that in suspension isomorphism, all the cup products go to zero, so there is no ring structure), but since Steenrod operations are compactible with suspension, it does have Steenrod operations, yet there is no relation between $Sq^*(X)$ and the dimension of X.

Recall that the cohomology of $H^*(K(\mathbb{Z}_2, n))$ is $\mathbb{Z}_2[Sq^I]$ for admissible I and excess $\leq n$. So the cohomology of the $H\mathbb{Z}_2$ spectrum, given by $\lim_{k\to\infty} H^{*+k}(K(\mathbb{Z}_2, k))$, is a $\mathbb{Z}_2$ vector space with basis Sq^I where I is admissible. In addition, since we can compose Steenrod operations, it has the structure of an algebra; it is called the Steenrod algebra. As a formal definition, the Steenrod algebra is the algebra over $\mathbb{Z}_2$ generated by formal symbols $Sq^1, Sq^2, \ldots$ modulo the Adem relation.

Remark 17. *How do we know that Adem relations eventually move any monomials to the reduced form? we can assign moments to I. In particular, if $I = (i_1, \ldots, i_n)$ then moment$(I) = \sum_{s=1}^{n} s i_s$. One can check that when you apply an Adem relation, the moment decreases.*

We shall eventually prove that two vector bundles are cobordant if and only if they have the same Stiefel-Whitney classes.

36 Describing MO

Recall the setup: we have $Gr_k(\mathbb{R}^\infty) = BO(k)$, and we write $\lim_{k\to\infty} Gr_k(\mathbb{R}^\infty) = BO$. We also have $MO = \{MO(k)\}$ where $MO(k) = \text{Thom}(BO(k); V_k)$. And we are interested in $\pi_n(MO) = \lim_{k\to\infty} \pi_{n+k}MO(k)$.

By Thom isomorphism, we know $H^*(MO(k)) = u\mathbb{Z}_2[w_1, \dots, w_k]$. Now recall the map induced by the Thom class $MO(k) \xrightarrow{u} K(\mathbb{Z}_2, k)$. Let's see what is not hit by pulling back along this map: ι goes to u, $Sq^1\iota$ hits uw_1, $Sq^2\iota$ hits uw_2, $Sq^3\iota$ hits uw_2, but note that we haven't hit uw_1^2. So there is in fact another thing, i.e. what we really have is $MO(k) \xrightarrow{u, uw_1^2} K(\mathbb{Z}_2, k) \times K(\mathbb{Z}_2, k+2)$. More generally, let $H\mathbb{Z}_2 = \{K(\mathbb{Z}_2, k)\}$ and write $X = \{K(\mathbb{Z}_2, k+2)\}$. Note that $\pi_i X = \mathbb{Z}_2$ where $i = 2$, and 0 otherwise, so we can write $X = \Sigma^2 H\mathbb{Z}_2$, using the fact that in spectra, $\pi_i X = \pi_{i+1}\Sigma X$. Using this language, we can write the map identified earlier as $MO \to H\mathbb{Z}_2 \vee \Sigma^2 H\mathbb{Z}_2$. This is again surjective, but we can obtain an isomorphism if we keep going:

Theorem 36.1. *$MO \cong \bigvee_{i=0}^{\infty} \Sigma^{s(i)} H\mathbb{Z}_2$ for some $\{s(i)\}_{i\geq 0}$, where by $\cong$ we mean (mod 2) weak equivalence.*

We will prove this by showing that $H^*(MO)$ is a free module over A, the Steenrod algebra.

Example 27. *Suppose X is a $(k-1)$-connected space, and suppose that $H^*(X;\mathbb{Z}_2)$ is free, finitely generated A-module through dimension $n < 2k$. Let $\{e_i\}$ be a (finite) basis, i.e. $e_i \in H^{|e_i|}(X;\mathbb{Z}_2)$. Now consider the map $X \to \prod K(\mathbb{Z}_2, |e_i|)$. In cohomology, we know that $H^*(K(\mathbb{Z}_2, |e_i|))$ is the module over admissible Is. In the range $* < n$ there is no product, so it is a vector space with basis $Sq^I \iota_{|e_i|}$, which is just $A\iota_{|e_i|}$. Now since the range also guarantees that there is no nontrivial contribution by the Kunneth formula, so $H^*(X) \leftarrow \bigoplus A\iota_{|e_i|}$ is an isomorphism for $* < n$, thus by mod C Hurewicz the map between associated spaces is a weak equivalence.*

We know that $A \to H^*(MO)$ given by $a \mapsto au$ is a monomorphism, but in fact there is some algebraic structure that we can play with. Here's a rough analogy: recall that if H is a subgroup of G, then one can consider the action of H on G (as a set), and then note that this action is free. What we have here, the freeness of $H^*(MO)$ over A, is something in the similar spirit. (The technically accurate statement is that "Steenrod algebra is the endomorphism ring of the Eilenberg-Maclane spectrum $H\mathbb{Z}_2$." Compare this with the statement of Theorem 36.1.)

Consider the canonical map $BO(k) \times BO(l) \to BO(k+l)$. This allows us to pull V_{k+l} back to $V_k \oplus V_l$. If we pass to the Thom complex, then we have the mapping $MO(k) \wedge MO(l) \to MK(k+l)$. Passing to homology gives $H_*MO \otimes H_*MO \to H_*MO$, making H_*MO into a commutative ring that is isomorphic to H_*BO via the Thom isomorphism. (The proof is a matter of bookkeeping.) Then we have an induced map, via the UCT, $H^*(MO) \otimes H^*(MO) \xleftarrow{\psi} H^*(MO)$, making it a coalgebra with a counit.

Definition 38. *A coalgebra over k is a vector space M equipped with a coproduct map $M \xrightarrow{\psi} M \otimes M$ and the counit $M \xrightarrow{\epsilon} k$, such that it satisfies the coassociativity: $1 \otimes \psi \circ \psi = \psi \otimes 1 \circ \psi$, and the counital law: $1 \otimes \epsilon \circ \psi = \epsilon \otimes 1 \circ \psi = id$.*

Note that Steenrod is also a coalgebra. Recall that we have $Sq^n = \sum_{i+j=n} Sq^i \otimes Sq^j$, so A is in fact a co-commutative coalgebra.

Claim 1. *The coproduct $A \xrightarrow{\psi} A \otimes A$ is a ring homomorphism.*

This makes the Steenrod algebra a Hopf algebra, which has both an algebra structure and a coalgebra structure. In general, a Hopf algebra can have an action $\cdot$ on a coalgebra, which is an action that is compactible with the coproducts, i.e. the following diagram commutes (where $\sigma_{2,3}$ swaps the 2nd and the 3rd factors):

$$\begin{tikzcd}
A \otimes M \arrow[rr, "\cdot"] \arrow[d, "\psi\otimes\psi"] & & M \arrow[d, "\psi"] \\
A \otimes A \otimes M \otimes M \arrow[d, "\sigma_{2,3}"] & & M \otimes M \\
A \otimes M \otimes A \otimes M \arrow[urr, "\cdot\otimes\cdot"'] & &
\end{tikzcd}$$

For example, A is a Hopf algebra, and $H^*(MO)$ is a coalgebra, and A acts on $H^*(MO)$ by the Cartan formula. Also note that everything we look at is graded and connected: $A = A_*, A_* = 0$ when $* < 0$, $A_0 = k$; $M = M_*$, $M_* = 0$ for $* < 0$, $M_0 = k$, $\psi 1 = 1 \otimes 1$.

Theorem 36.2 (Milnor-Moore)**.** *If A is a connected graded Hopf-algebra, acting on a connected graded coalgebra M, where $1 \in M_0$ is sent to $1 \otimes 1$ by ψ, and if $a \mapsto a \cdot 1$ is a monomorphism, then M is free over A.*

Proof. First some side observations. A maps to k by the counit ϵ. (In the case of Steenrod algebra, $\ker \epsilon$ is generated by all the squares.) Now suppose $M = \bigoplus A$, then $k \otimes_A M = \bigoplus k \otimes_A A = \bigoplus k$. This tells us where the generators of M in terms of A are. This motivates looking at $k \otimes_A M$.

Let $\overline{M} = k \otimes_A M$. Choose a vector space section $\sigma : \overline{M} \to M$, such that its concatenation with $M \to \overline{M}$ is id, then there is a mapping $A \otimes \overline{M} \to M$ given by $a \otimes x \mapsto a \cdot x$ which is a map of left A-modules. The claim is that this map is an isomorphism with our assumptions. First we need to check it's an epimorphism. This is fairly straightforward via a proof by induction on the grading. (Kind of like Nakayama's lemma.) Now to check it's mono, look at the following: $A \otimes \overline{M} \to M \xrightarrow{\psi} M \otimes M \to M \otimes \overline{M}$ (all tensored over k), and show this entire thing is a mono. The key step is to define $F_i = \overline{M}_{* \leq i} \subseteq \overline{M}$. The big map above induces $A \otimes F_i \to M \otimes F_i$. By induction on i, we assume mono for degree up to $i-1$. Now consider the LES $0 \to A \otimes F_{i-1} \to M \otimes F_i \to M \otimes F_i / F_{i-1} \to 0$, and note that the last one is given by $a \otimes x \mapsto a \cdot 1 \otimes x$, so it is mono; and the first one is mono by assumption. $\square$

37 Finishing the Cobordism Story

We showed that the cobordism ring is the same as the homotopy groups of the MO spectrum, and that the cohomology of MO is free over the Steenrod algebra, i.e. $MO \cong \bigvee \Sigma^* H\mathbb{Z}_2$ (or, equivalently, $MO(k)$ is equivalent to a product of $K(\mathbb{Z}_2, j)$ for some js through dimension $2k$).

Corollary 17. *The Hurewicz homomorphism $\pi_n(MO) \to H_n(MO)$ is a monomorphism since it is true for $H\mathbb{Z}_2$. In fact, $\pi_*(MO) = Hom_A(H^*(MO), \mathbb{Z}_2)$.*

So how big is $\pi_n(MO)$? Consider the Poincare series $P(MO, t) = \sum \dim \pi_k(MO) t^k$; we also have $P(H_*MO, t) = \sum \dim H_k(MO) t^k = \sum \dim H^k(MO) t^k$. Since $MO = \bigvee_S \Sigma^{|S|} H\mathbb{Z}_2$, we know that $\pi_*(MO)$ has basis S over $\mathbb{Z}_2$, and $H^*(MO) = \mathbb{Z}_2[S] \otimes H^*(H\mathbb{Z}_2) = \mathbb{Z}_2[S] \otimes A$. In other words, $P(H_*MO) = P(A)P(\pi_*MO)$. We know that $H^*(MO) = H^*(BO)$ by Thom isomorphism, which is just $\mathbb{Z}_2[w_1, w_2, \ldots]$, thus

$$P(H^*MO) = \frac{1}{(1-t)(1-t^2)(1-t^3)\ldots}.$$

On the other hand, a basis of admissible monomoials gives that

$$P(A) = \frac{1}{(1-t)(1-t^3)(1-t^7)\ldots(1-t^{2^n-1})}.$$

Thus we know that

$$P(\pi_*MO) = \prod_{n \neq 2^j - 1} \frac{1}{1-t^n}.$$

Our goal today is the following theorem, which says the structure of $\pi_*(MO)$ is exactly the one predicted by the Poincare series.

Theorem 37.1. *$\pi_n(MO)$ is $\mathbb{Z}_2[x_n \mid n > 0, n \neq 2^j - 1]$.*

Observe that we have $\pi_*(MO) \hookrightarrow H_*(MO) = H_*(BO)$. So what is $H_*(BO)$? Recall the mapping $\mathbb{Z}_2[w_1, \ldots, w_k] = H^*(BO(k)) \to H^*((\mathbb{RP}^\infty)^k) = \mathbb{Z}[x_1, \ldots, x_k]$, where $x_i = w_i(L_i)$. In this map, $w_n \mapsto \sigma(x_1, \ldots, x_k)$ where $\prod_i (1 + tx_i) = \sum \sigma_n t^n$ (σ_n are the symmetric functions). In fact, $H^*(BO(k))$ is the ring of symmetric invariants in $\mathbb{Z}[x_1, \ldots, Z_k]$.

Corollary 18. *Write $\tilde{H}_k(\mathbb{RP}^\infty) = H_*(\mathbb{RP}^\infty)^{\otimes k}$, then $Sym(\tilde{H}_k \mathbb{CP}^\infty) \to H_*(BO)$ is surjective. However, as we look at the Poincare series, we can conclude that it is in fact an isomorphism.*

Proof. Choose $b_i \in H_*\mathbb{RP}^\infty$ such that $\langle x^j, b_i \rangle = \delta_{ij}$ (the Kronecker pairing on cohomology, where x is the generator of $H_*\mathbb{RP}^\infty$), then the former is $\mathbb{Z}_2[b_1, b_2, \ldots]$. Now compare the Poincare series. □

Lemma 14. *Suppose $b_i' \in H_i BO$ are elements satisfying $b_i' = b_i \mod I^2$ where $I = (b_1, b_2, \ldots)$, then the map from $\mathbb{Z}_2[b_1', b_2', \ldots]$ to H_*BO is an isomorphism.*

Let $s_n = s_n(w_1, w_2, \ldots)$ be the symmetric function given by the Newton polynomials. In other words, they are functions such that $s_p(\sigma_1, \ldots, \sigma_n) = (-1)^{p+1} \sum_{k=1}^{n} x_k^p$, where σ_i are the elementary symmetric functions. Then one can see that $s_n \in H^n(BO)$. (One should think of them as characteristic classes of vector bundles.)

Proposition 38. *$s_n(V \oplus W) = s_n(V) + s_n(W)$.*

Proof. Follows from the splitting principle. □

Proposition 39. *Equivalently, under the coproduct map $H^*(BO) \to H^*(BO) \otimes H^*(BO)$, we have $s_n \mapsto s_n \otimes 1 + 1 \otimes s_n$. An element having such property is called a primitive element.*

Example 28. *If L is the tautological bundle over $\mathbb{RP}^\infty$, then $s_n(L) = x^n$. In cohomology, this says that $s_n \in H^*(BO)$ pulls back to x^n in $H^n(\mathbb{RP}^\infty)$. Let $i : \mathbb{RP}^\infty \to BO$, then $b_n \mapsto i_* b_n$, the latter of which we sometimes also write as b_n. Then $\langle s_n, b_n \rangle = \langle s_n, i_* b_n \rangle = \langle i^* s_n, b_n \rangle = \langle x^n, b_n \rangle = 1$.*

Corollary 19. *$\langle s_n, b_m \rangle = \delta_{mn}$.*

Proposition 40. *Suppose $a \in H_k(BO)$ and $b \in H_l(MO)$ such that $k, l > 0, k + l = n$, then $\langle s_n, ab \rangle = 0$.*

Proof. $\langle \psi(a \otimes b) \rangle = \langle \psi(s_n), a \otimes b \rangle = \langle s_n \otimes 1 + 1 \otimes s_n, a \otimes b \rangle = 0$ (consider degree). ☐

Theorem 37.2. *$b'_n \in H_n(BO)$ satisfies $b'_n = b_n$ mod I^2 iff $\langle s_n, b'_n \rangle = 1$.*

Corollary 20. *$\{b'_n\}$ are polynomial generators iff $\langle s_n, b'_n \rangle = 1$.*

Now consider the Hurewicz homomorphism $\pi_k(MO) \to H_k(MO)$. Consider a manifold M of dimension k, and consider $[M]$ representing M in $\pi_k(MO)$[17], then it maps to $H([M^k]) \in H_k(MO) = H_k(BO)$.

Theorem 37.3. *Given $w \in H^*(BO)$, then $\langle w, H([M^k]) \rangle = \langle w(\nu), [M] \rangle$, where ν is the normal bundle of an embedding $M \subseteq \mathbb{R}^N$.*

We are interested in $\langle s_k, H([M]) \rangle = \langle s_k(\nu), [M] \rangle$. But note that $\nu + TM$ is the trivial bundle of dimension N, thus $s_k(\nu) + s_k(TM) = 0$, thus $s_k(v) = s_k(TM)$ (mod 2); hence we conclude that

Theorem 37.4. *If M_i for $i \neq 2^n - 1, i > 0$ is a sequence of manifolds satisfying $\langle s_i(TM), [M] \rangle = 1$, then $\pi_*(MO) = \mathbb{Z}_2[\ [M_i]\ ]$.*

Proof. We just need to prove that we have a polynomial algebra of the right size. The elements $b'_i = H([M_i]), i \neq 2^n - 1$, and the elements b_{2^n-1} form the polynomial algebra generators for $H_*(MO)$, thus $\mathbb{Z}_2[\ [M_i]\ ] \to \pi_*(MO) \to H_*(MO)$ is a monomorphism with the same Poincare series. ☐

So what are the generators, explicitly? Recall that $T\mathbb{RP}^n = (n+1)L$, thus $s_n(T\mathbb{RP}^n) = (n+1)x^n$, thus $\langle s_n T(\mathbb{RP}^n), [\mathbb{RP}^n] \rangle = n + 1$, so we have half of the generators (if n is even). For a while it was a mystery what the odd generators are.

First construction was given by Dold by considering the $\mathbb{Z}_2$ action on $S^1 \otimes \mathbb{CP}^m / \{(\lambda, t) \sim (\lambda^{-1}, \bar{t})\}$. The second one given by Milnor, by considering certain hypersurfaces (called Milnor hypersurfaces) of degree 2, denoted by $M \subseteq \mathbb{RP}^a \times \mathbb{RP}^b$. For instance, we can let M be the zero set $\displaystyle\sum_{i \leq \min(a,b)} x_i y_i$. Let $i : M \hookrightarrow \mathbb{RP}^a \times \mathbb{RP}^b$ be the embedding, and ν the normal bundle. The RHS has a line bundle $L_1 \otimes L_2$, let $x = w_1(L_1), y = w_1(L_2)$, then $[M] = x + y$. Consider $TM + i^*(L_1 \otimes L_2) = i^*T(\mathbb{RP}^a \times \mathbb{RP}^b)$. Suppose $a, b > 1$ and $a + b = n - 1$. Then $s_n(TM) = i^*((x+y)^n) - i^*((a+1)x^n + (b+1)y^n)$, and $\langle s_n(TM), [M] \rangle = \langle i^*(x+y)^n, [M] \rangle = \langle (x+y)^{n-1}, [\mathbb{RP}^a \times \mathbb{RP}^b] \rangle = \binom{n+1}{a}$ (mod 2). If n odd and $n \neq 2^k - 1$, then $n + 1 = 2^k(2a + 1) = 2^{k+1}q + 2^k$ for some $q > 0$; write this as $a + b$, then $\binom{a+b}{a} = 1$. Thus we know that we have obtained all the generators. This completes the classification of manifolds up to cobordism.

Example 29. *If one has a complex variety defined over the reals, and consider the fixed points under complex conjugation of the variety, then up to cobordism, the complex variety is the square of the fixed points. For example, $\mathbb{CP}^n \cong_{cb} (\mathbb{RP}^n)^2$.*

[17] Careful: $[M]$ is also sometimes used to refer to the fundamental class in $H_k M$.