18.966 Differential Topology Lecture Notes

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This class was taught by Prof. Paul Seidel in the Spring term of 2016 at MIT. The following notes were taken by Yuchen Fu. They have not been very carefully proofread and all errors are the notetakers'. The daily homework problems are included under the tag "problem."

Contents

1 Feb 03

Definition 1. *Let $U, V \subset \mathbb{R}^n$ be open subsets. A smooth map $f : U \to V$ is a diffeomorphism if it has a smooth inverse.*

Example 1. $(-1, 1)$ *and* $\mathbb{R}$ *are diffeomorphic subsets of* $\mathbb{R}$ *by the map* $\tan(\pi x/2)$. *On the other hand,* $x^3 : \mathbb{R} \to \mathbb{R}$ *is not a diffeomorphism.*

Theorem 1.1 (The Inverse Mapping Theorem)**.** *Let $U, V \in \mathbb{R}^n$, $f : U \to V$ such that $x_0 \in U, y_0 = f(x_0) \in V$. If $Df_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, then there are open subsets $\tilde{U}, \tilde{V}$ such that $f|_{\tilde{U}} : \tilde{U} \to \tilde{V}$ is a diffeomorphism.*

Note that in order to prove this, the inductive procedure that produces the Taylor expansion only gives the Taylor series, which doesn't necessarily give the inverse function. Thus we need a better control (e.g. using contraction mapping theorem).

Question 1. *Is the inverse mapping theorem true in the world of holomorphic maps?*

The answer is yes. One can first prove that the real inverse is holomorphic. Or one can use locally convergent Taylor series.

Question 2. *Is the inverse mapping theorem true for complex polynomials where "open subset" is true in the Zariski topology?*

No. Consider $\mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ given by z^2. It sends 1 to 1. The inverse (square root) is not going to be a polynomial. This problem can be dealt with by introducing the etale topology.

Theorem 1.2 (Local Submersion)**.** *Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$. Suppose $0 \mapsto 0$ by $f : U \to V$. Further, suppose that Df_0 is onto, then after shrinking U, there is a diffeomorphism $\varphi : \tilde{U} \to U$ (where $\tilde{U} \ni 0$) mapping 0 to 0 such that $f \circ \varphi$ is a (surjective) linear map.*

Note This is usually written in a more technical form as the implicit function theorem.

Note that this in particular shows that Df was surjective in a neighborhood; of course this is no problem since being surjective is an open condition. Do we have an injective counterpart? Yes.

Theorem 1.3 (Local Immersion)**.** *Same situation as above. Suppose that Df_0 is injective, then after shrinking U and V, there is a diffeomorphism $\psi : V \to \tilde{V}$ mapping 0 to 0 such that $\psi \circ f$ is an (injective) linear map.*

Both are proven by reducing to the inverse mapping theorem by adding extra dimensions to source / target.

And here's a more generalized version, which answers the question "when does a function look like a linear one?"

Theorem 1.4 (Theorem of Constant Rank)**.** *Same situation as above. Suppose that* $\text{rank} Df_x$ *is constant. Then after shrinking U and V, there are diffeomorphisms $\varphi : \tilde{U} \to U, \psi : V \to \tilde{V}$ (both preserving 0) such that $\psi \circ f \circ \varphi$ is a linear map.*

Definition 2. *Smooth $f : U \to V$. We say that $y \in V$ is a regular value of f if Df_x is onto for all $x \mapsto y$.*

Definition 3. *f is a submersion of Df_x is onto for all $x \to U$, i.e. all values are regular values.*

Definition 4. *f is an immersion if Df_x is injective for all $x \in U$.*

Definition 5. *If f is an immersion, injective and proper, then it's call an embedding.*

Note Proper is as a map $U \to V$. We define it as if (x_i) doesn't have a convergent subsequence, then $(f(x_i))$ also doesn't have a convergent subsequence. The image of an embedding is a closed subset of V.

Example 2. *Take $f : (0, 2\pi) \to \mathbb{C}$, $f(t) = e^{it}$. This is not an embedding (it breaks properness). However, $f : (0, \pi) \to \{z \mid \mathrm{im}(z) > 0\}$ by $f(t) = e^{it}$ is an embedding.*

Definition 6. *Let $U \subset \mathbb{R}^n$ be an open subset. A closed subset $M \subseteq U$ is called a submanifold if around each point of M there exists a local diffeomorphism which transforms M into a linear subspace.*

Corollary 1. *If y_0 is a regular value of $f : U \to V$ then the preimage of y_0 is a submanifold.*

Of course, if f is a submersion, then this is true for all y_0.

Note $\dim(f^{-1}(y_0)) = m - n$ where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$.

Corollary 2. *If $f : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ is an embedding, then $f(U) \subset V$ is a submanifold of dimension m.*

A general strategy to prove something is a submanifold is either showing it's a preimage or showing that it's an image.

Note Linear maps can always be factored into a surjection followed by an injection (take the middle space to be the image). Correspondingly, a map of constant rank can be locally factored into a submersion followed by an immersion.

Here's an application.

Theorem 1.5 (Local Retraction)**.** *Let $U \subset \mathbb{R}^n$ be an open subset, and $f : U \to U$ a smooth map with $f(f(x)) = f(x)$ (it's called a retraction), then $f(U)$ is a submanifold of U.*

Proof. We have $Df_{f(x)} \circ Df_x = Df_x$. In particular, if $x \in f(U)$, Df_x is a projection. Projections have only eigenvalues 0 and 1. By continuity of eigenvalues, they are locally constant on $f(U)$. But the image is not an open subset, so we can't apply the constant rank theorem yet.

For general $x \in U$, $\mathrm{rank} Df_x = \mathrm{rank}(Df_{f(x)} \circ Df_x) \leq \mathrm{rank} Df_{f(x)}$. On the other hand, if x is sufficiently closed to $f(x)$, then $\mathrm{rank} Df_x \geq \mathrm{rank} Df_{f(x)}$ due to semicontinuity of ranks. This implies that $\mathrm{rank} Df_x = \mathrm{rank} Df_{f(x)}$ if x is sufficiently close to $f(x)$. Thuse in a neighborhood of $f(U)$, $\mathrm{rank} Df$ is locally constant.

Now apply theorem of constant rank to make our conclusion. □

That was also another way of showing that things are submanifolds.

Problem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(0) = 0, Df_0 = 0$, $D^2 f$ is everywhere positive definite. Prove that $f^{-1}(y)$ for $y > 0$ is a submanifold of dimension $n - 1$ (show they are regular values) (some convexity property).*

Problem 2. *Let $U \xrightarrow{f} U$ where $0 \mapsto 0$ be a smooth map such that $f(0) = 0, f(f(\ldots(f(x)))\ldots) = x$ (pth order) for all x. Find a local diffeomorphism ψ such that $\psi \circ f \circ \psi^{-1}$ is linear. (Applying inverse function theorem won't work). (ψ is tricky to construct. Try to average things.)*

2 Feb 08

Regarding the problem 2 from last time. First thing to notice that $\psi \circ f \circ \psi^{-1}$ is of finite order p. (This is the start of study of nonlinear symmetry. It in fact holds for any finite group or even compact Lie group.) Also, it conjugates to Df_0 (since $\psi \circ f \circ \psi^{-1} = D(\psi f \psi^{-1})_0 = D\psi_0 \circ Df_0 \circ D\psi_0^{-1}$), so we actually aim to make $\psi \circ f \circ \psi^{-1} = Df_0$. so what we want is $\psi \circ f = Df_0 \circ \psi$. Now we need to use the finite order property. (This won't hold in general because the dynamic of a fixed point is more complicated than a linear map, although things like Hartman-Grobman theorem says that there are some things to be said.)

$\psi(x) = x$ won't satisfy this, but it satisfies up to first order (derivative is okay). What about $Df_0(f^{-1}(x))$? It's just as equally good. Now there are p different choices $Df_0^i(f^{-i}(x))$. Now of course each gets sent to next by $Df_0 \circ f^{-1}$, so we take $\psi(x) = 1/p(x + Df_0(f^{-1}(x))) + \ldots + Df_0^{p-1}(f^{1-p}(x)))$. Then $Df_0(\psi(x)) = 1/p(Df_0(x) + Df_0^2(f^{-1}(x)) + \ldots + f(x))$, and $\psi(f(x)) = 1/p(f(x) + Df_0(x) + \ldots + Df_0^{p-1}(f^{2-p}(x)))$.

It remains to check that ψ is a local diffeomorphism by applying the inverse function theorem. $D\psi_0 = 1/p(I + Df_0 Df_0^{-1} + \ldots) = I$.

Of course this extends to complex geometry (nothing about real is used), but of course if your characteristic is p then we're not good.

Now let's go back to schedule. Recall this from last lecture: Let $U \subseteq \mathbb{R}^m$ open, $f : U \to U$ satisfies $f(f(x)) = f(x)$, then $M = f(U)$ is a submanifold.

Example 3. *Every matrix $A \in \mathrm{GL}_n(\mathbb{R})$ can be uniquely written as $A = PQ$ where P is symmetric positive definite, and Q is orthogonal.*

In representation theory this is generalized by the Cartan decomposition. Note that $AA^t = PQQ^tP^t = PP^t = P^2 \implies P = (AA^t)^{1/2}$, and correspondingly, $Q = P^{-1}A$.

Taking the square root of a positive definite matrix is a differentiable (in fact smooth) (with respect to its coefficients) operation.

Three proofs:

1. Squaring is a differentiable thing and we invert it. Boring.
2. $p(A)$ for p is a polynomial is certainly differentiable. Fix A_0, Choose p to approximate square root to high order at the eigenvalues of A_0. Then $p(A) \approx \sqrt{A}$ for $A \approx A_0$ to high order.
3. Functional calculus (Cool stuff). Remember for $\lambda > 0$, $\lambda^{1/2} = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-\lambda} z^{1/2}$. Similarly, if γ contains all eigenvalues of A, then $A^{1/2} = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-A} z^{1/2}$.

   (To see this, suppose A is diagonal, then easy to see this is true (integrate on each entry). But this is independent of basis, so this must be true in general.)

   But the right hand side is clearly differentiable in A.

Hence $f(A) = (AA^t)^{1/2}A$ is a differentiable map, and $f(f(A)) = f(Q) = f(A)$. Hence $f(\mathrm{GL}_n(\mathbb{R})) = \mathrm{O}_n(\mathbb{R})$ is a submanifold of $\mathrm{GL}_n(\mathbb{R})$. (Since it's closed in $\mathbb{R}^{n^2}$, it is also a submanifold of that.) Another proof is to consider the orthogonal group as the preimage of $f(X) = XX^t$ at I and apply the implicit function theorem.

And here is an converse:

Application Let $U \subseteq \mathbb{R}^m$ open, $M \subseteq U$ a submanifold. Then there is an open $M \subseteq \tilde{U} \subseteq U$ and a smooth map $f : \tilde{U} \to \tilde{U}$ such that $f(f(x)) = f(x), f(\tilde{U}) = M$. Locally this is no problem: we just project to the linear subspace. But how does this make sense globally?

Definition 7. *Let $M \subseteq U \subseteq \mathbb{R}^m$ be a submanifold. The tangent space of M at $x_0 \in M$ is the linear subspace $TM_{x_0} \subseteq \mathbb{R}^m$ defined as follows: $TM_{x_0} = \{\frac{dc}{dt}|_{t=0} \mid c : (-\epsilon, \epsilon) \to \mathbb{R}^m, c(0) = x_0, c(t) \in M \ \forall t\}$.*

It's not immediately obvious that this is a linear space; this is actually a feature, because this also works with objects with singularities (we then get something called tangent cones), in which case they are *not* linear spaces.

A second definition goes as follows: $TM_{x_0} = \{D\varphi_0(\mathbb{R}^n \times \{0\}) \mid \varphi : \mathbb{R}^m \to \mathbb{R}^m \text{ isalocaldiffeomorphism}, \varphi(0) = x_0, \mathbb{R}^n \times \{0\} \mapsto M \text{ locally}\}$. It's clearly a linear space, but not clear it's well-defined (it seems to depends on a choice of φ).

Third definition (Zariski tangent space):

$$TM_{x_0} = \{X \subseteq \mathbb{R}^m \mid \text{if } f \text{ is a function defined near } x_0 \text{ and which vanishes on } M, \text{ then } Df_{x_0}(X) = 0\}.$$

All three are equivalent in the case of manifolds. In general they are different.

Example 4. *If $f : U \to V$, $y \in V$ a regular value, $M = f^{-1}(y) \ni x$, then $TM_x = \ker(Df_x)$.*

Example 5. *If $f : U \to V$ is an embedding, $M = f(U)$ and $y = f(x)$, then $TM_y = \mathrm{im}(Df_x)$.*

Now we return to the question of retractions.

Lemma 1 (Orthogonal Retraction)**.** *If $M \subseteq \mathbb{R}^m$ is a submanifold, $x_0 \in M$, then there exists small open subsets $x_0 \in U \subseteq \tilde{U} \subseteq \mathbb{R}^m$ such that for each $x \in U$ there exists a unique $\tilde{x} \in M \cap \tilde{U}$ such that $x - \tilde{x} \perp TM_{\tilde{x}}$.*

This implies the existence of global orthogonal retraction, as the map to $\tilde{x}$ is unique.

Proof. Suppose $\dim(M) = n$. Find a local parametrization i.e. some $0 \in V \subseteq \mathbb{R}^n$ open and $g : V \xrightarrow{\text{immersion}} \mathbb{R}^m$, $g(0) = x_0, g(V) = M$ locally near x_0. Possibly after shrinking V we can find $\xi_1, \dots, \xi_{m-n} : V \to \mathbb{R}^n$ such that $\xi_1(y), \dots, \xi_{m-n}(y)$ is a basis for $TM^{\perp}_{g(y)}$. Then define $\tilde{g} : V \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ with $\tilde{g}(y,t) = g(y) + \sum_i t_i \xi_i(y)$. Then $\tilde{g}$ is a local diffeomorphism by the inverse mapping theorem. So if $x = \tilde{g}(y,t)$ then set $\tilde{x} = \tilde{g}(y, 0)$. □

Definition 8. *Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ be open, $N \subseteq V$ a submanifold, and $f : U \to V$ a smooth map. We say that f is transverse to N if for all $y \in N$, $x \in f^{-1}(y)$, $\mathrm{im}(Df_x) + TN_y = \mathbb{R}^n$.*

Example 6. *If $m + \dim(N) < n$, then transversality says that $f(U) \cap N =$. If $N = \{y\}$, transversality means y is a regular value of f. If $N = V$, transversality is automatic. If f is a submersion, it's transverse to everything.*

Problem 3. *If f is transverse to N, then $M = f^{-1}(N)$ is a submanifold of U.*

3 Feb 10

(Something about Grassmannian. Didn't write it down.)

Lemma 2 (Reduction)**.** *Take $0 \in U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$ smooth, $f(0) = 0$ and $\mathrm{rank}\, Df_0 = r$, then there are local diffeomorphisms φ, ψ around 0 such that $(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_r, \text{somethingelse})$ and $(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_r, 0, \dots, 0) = (x_1, \dots, x_r, 0, \dots, 0)$.*

See proof of constant rank theorem in the textbook.

Now let's talk about Banach spaces. There is a notion of differentialbility (smoothness) for maps between (open subsets of) Banach spaces

$H \supseteq U \xrightarrow{f} K$ where H, K Banach. The derivative $Df_x : H \to K$ is a (bounded) linear operator.

The **inverse mapping theorem** still holds.

Local submersion theorem: $H \supseteq U \ni x_0 \xrightarrow{f} K$, and that there is a decomposition $H = H_1 \oplus H_2$ such that $Df_{x_0}|_{H_1} : H_1 \to K$ is an isomorphism. Then there is a diffeomorphism φ near x_0 so that $f \circ \varphi$ is a (surjective) linear operator.

The seemingly complicated definition roots from the fact that a subspace in a Banach space needs not have complement.

Remark 1. *If H is a Hilbert space, it is enough to ask that Df_{x_0} is onto (then $H_2 = \ker(Df_{x_0}), H_1 = \ker(Df_{x_0})^{\perp}$).*

Local immersion theorem $H \supseteq U \ni x_0 \xrightarrow{f} K$, and there is $K = K_1 \oplus K_2$ such that $H \xrightarrow{Df_{x_0}} K \xrightarrow{\text{project}} K_1$ is an isomorphism, then there is ψ near $f(x_0)$ so that $\psi \circ f$ is a linear operator.

Remark 2. *If K is a Hilbert space, it is enough to ask Df_{x_0} is injective with closed image.*

Theorem of Constant Rank $H \supseteq U \xrightarrow{f} K$ and there exists $H = H_1 \oplus \ker(Df_{x_0}), K = K_1 \oplus K_2$ such that $H_1 \hookrightarrow H \xrightarrow{Df_x} K \twoheadrightarrow K_1$ is an isomorphism for x close to x_0. Then there are ψ, φ such that $\psi \circ f \circ \varphi$ is a linear map.

Definition 9. *Let H, K be Banach spaces. An operator $A : H \to K$ is Fredholm if $\ker(A) \subseteq H$ is finite-dimensional, and $\mathrm{im}(A) \subseteq K$ is finite-codimensional (it is then always closed – note that it's not automatically true for any finite-codimenionsal space; it is special as the image of a linear operator).*

Theorem 3.1. *A is Fredholm if and only if there is $B : K \to H$ such that $BA = I + \text{compactoperator}$ and $AB = I + \text{compactoperator}$.*

It says that A only fails to be invertible by something compact.

Example 7. *If A_1 is invertible, A_2 is compact, then $A_1 + A_2$ is Fredholm. NOT every Fredholm operator is of this form.*

Definition 10. *The Fredholm index of A is defined by $\dim(\ker(A)) - \operatorname{codim}(\operatorname{im}(A))$.*

The index is locally constant in the space of Fredholm operators.

Example 8. *A_1 invertible, A_2 compact, then $A_1 + A_2$ has Fredholm index 0. (Consider the family $A_1 + \varepsilon A_2$ that deforms this to A_1.)*

Corollary 3. *Let $U \to K$ be a map whose derivatives are Fredholm (a "Fredholm map"). If $y \in K$ is a regular value, then $f^{-1}(y)$ is a finite dimensional submanifold whose dimension equals index of Df.*

Problem 4. *Let H be a separable Hilber space. Take a map $f : H \to H$ given by $f(x) = x - g(Ax)$ where g is a smooth bounded map, A is a compact linear operator. Prove that if y is a regular value, then $f^{-1}(y)$ a finite set. If $g = 0$ then we clearly have one solution. So this is a pertubation of $x = y$. (finite = bounded and compact as a zero-dimensional subspace.)*

4 Feb 15

Definition 11. *A subset $A \subseteq \mathbb{R}^n$ has measure 0 if, for every $\varepsilon > 0$, there exists a countable collection R_i of cubes/balls/parallelegrams/etc such that $A \subseteq \bigcup R_i$, $\sum_i \operatorname{vol}(R_i) < \varepsilon$.*

A countable union of measure zero subsets is a measure zero subset.

Lemma 3. *$U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ smooth. If $A \subseteq U$ is a measure zero subset, then so is $f(A)$.*

Proof. It's enough to show this for $A \subseteq K_1 \subseteq \operatorname{int}(K_2) \subseteq K_2 \subseteq U$ is compact (since any A can be written as a countable union of such subsets). Any sufficiently small ball centered in K_1 lies in K_2. Since $\|Df_x\| \le C$ for all $x \in K_2$, the image of such a ball of radius ρ lies inside a ball of radius $C\rho$. Cover A with balls of total volume $\le \varepsilon \implies$ image is covered by balls of total volume $\le \varepsilon C^m$. □

Theorem 4.1. *$\mathbb{R}^n$ does not have measure 0.*

This implies that no ball (or open subset) has measure zero. In other words, measure zero subsets can't have interior points.

Definition 12. *A subset $B \subseteq \mathbb{R}^n$ is called thin in the sense of Baire (a.k.a. first Baire category) if it can be written as $B = \bigcup_i C_i$, where the C_i are closed subsets with no interior points.*

A countable union of thin subsets is a thin subset.

Theorem 4.2 (Baire)**.** *$\mathbb{R}^n$ is not thin.*

This implies that thin sets cannot have interior points.

Remark 3. *If A has measure zero and is a countable union of closed subsets, then A is thin.*

Remark 4. *One can write $\mathbb{R}^n = A \cup B$, where A has measure 0 and B is thin.*

Theorem 4.3 (Sard)**.** *$U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$ smooth, then the set of critical values of f has measure 0.*

Corollary 4. *The set of critical values is also thin in the sense of Baire. Why?*

Write $\text{Crit}(f) = \{x \in U : Df_x \text{ is not onto}\}$, $\text{CritV}(f) = f(\text{Crit}(f))$, then $\text{Crit}(f)$ is closed, hence a countable union of compact subsets, so $\text{CirtV}(f)$ is a countable union of closed subsets.

And the following is the infinite version.

Theorem 4.4 (Sard-Smale)**.** *Take H, K Banach spaces which are second countable (have countable dense subsets), $U \subset H$ open, $f : U \to K$ smooth Fredholm map. Then the set of critical values is thin in the sense of Baire.*

Therefore regular values are dense in this case.

Special cases of Sard's theorem:

Lemma 4 (Trivial Sard)**.** *$U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$, $m < n$, then $f(U)$ has measure 0 (as one cannot have any regular value).*

Proof. (Sketch of Proof) Similar to the previous lemma. Suppose U is bounded and $\|Df\|$ bounded, then cover it by $\sim N^m$ cubes of size $\sim 1/N$, then the image is covered by $\sim N^m$ cubes of size $\sim 1/N$, but their total volume is $\sim 1/N^{n-m}$. (A bit more detail: consider Taylor's expansion near the critical point. See here for detail.) □

Lemma 5 (Easy Sard)**.** *$U \subseteq \mathbb{R}^m$, $f : U \to \mathbb{R}^m$ smooth, then the set of singular values has measure zero.*

Proof. Same idea except at the critical values $\text{Det}(Df) = 0$, hence cubes nearby get shrunk to a small amount. In particular, suppose we have some cover by small squares, then near the singular points, these squares get shrunk arbitrarily small. □

The difficult case of Sard's theorem is where the source dimension is larger than the target dimension, and it in fact uses higher derivatives.

Theorem 4.5. *$p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, then p has only finitely many critical values.*

Theorem 4.6. *$p : \mathbb{R}^n \to \mathbb{R}$ is a real analytic function, then p has at most countably many critical values.*

Example 9. *$p : \mathbb{R}^n \to \mathbb{R}$ polynomial, proper, bounded below, then level sets $p^{-1}(y)$ are compact hypersurfaces for all $y \gg 0$.*

Roughly speaking, these level sets will all have the same topology.

Proof. $\text{Crit}(p) \subseteq \mathbb{R}^n$ is defined by polynomial equations $(\partial P/\partial x_i = 0)$ for each i, hence has only finitely many connected components.

Second fact (Milnor's curve selection lemma): Let $X \subseteq \mathbb{R}^n$ be a semialgebraic subset (defined by polynomial equalities and inequalities). For any $x_0 \in \overline{X} \backslash X$, there is a smooth curve $c : [0, \varepsilon) \to \mathbb{R}^n$ such that $c(0) = x_0, c(t) \in X$ for all $t > 0$.

Now take $x_0 \in \text{Crit}(p)$, $X = \{x \in \text{Crit}(p) \mid f(x) \neq f(x_0)\}$. Then either $x_0 \notin \overline{X}$, which means $f|_{\text{Crit}(p)}$ is locally constant near x_0, otherwise by the curve selection lemma, there exists some c such that $c(0) = x_0$, $c(t) \in \text{Crit}(p)$ for all t, $p(c(t)) \neq p(c(0))$ for all $t > 0$, but this is a contradiction, because $\frac{d}{dt}(p(c(t))) = D_p(\frac{dc}{dt}) = 0$, since $c(t)$ is always critical. □

5 Feb 16

Theorem 5.1 (Sard)**.** *$U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$ smooth, then the set of critical points has measure zero.*

Proof. Let $D = \text{Crit}(f)$, $D_1 = \{x \in U : Df_x = 0\}$, $D_2 = \{x \in U : \text{derivatives of order less than 2 vanish}\}$.

First let's prove that $f(D \backslash D_1)$ has measure zero. Take $x \in D \backslash D_1$, let's assume for simplicity that $(Df_{\overline{x}})_{11} = (\partial_1 f_1)_{\overline{x}} \neq 0$. Consider $h(x) = (f_1(x), x_2, \ldots, x_m)$, then Dh is invertible at $\overline{x}$, so this is a local diffeomorphism. Let's take a local inverse φ, then $(f \circ \varphi)(x) = (x_1, g(x_1, \ldots, x_m))$. Therefore the first column of $D(f \circ \varphi)$ is $(1, 0, \ldots, 0)$. Critical points of $f \circ \varphi$ are then critical points of g where x_1 is considered fixed. Hence critical values of $f \circ \varphi$ equals $\coprod_{x_1} \{x_1\} \times$ (critical values of g with x_1 considered fixed). By induction on m, the set of critical values of g with x_1 fixed has measure zero in $\mathbb{R}^{n-1}$. By Funibi's theorem, critical values of $f \circ \varphi$ has measure zero. Everything until now is still local, but we can cover $D \backslash D_1$ with countably many balls, so the global statement follows.

Now let's show that $f(D_k \backslash D_{k+1})$ has measure zero. Fix $\overline{x} \in D_k \backslash D_{k+1}$, we may assume $\dfrac{\partial^{k+1} f_1}{\partial x_1 \partial x_{i_2} \dots \partial x_{i_{k+1}}} \neq 0$ at $\overline{x}$. Consider $h(x) = \left(\dfrac{\partial^k f_1}{\partial x_{i_2} \dots \partial x_{i_{k+1}}}, x_2, \dots, x_m \right)$. This is a local diffeomorphism, hence has a local inverse φ. Then each point of $f(D_k)$ locally near $\overline{x}$ is a critical value of $f \circ \varphi|_{\{0\} \times \mathbb{R}^{m-1}}$. By induction on m, this has measure zero.

Finally, if k is sufficiently large $k > m/n$, then $f(D_k)$ has measure zero. Sketch of proof: in U closed to D_k, we use cube covering it, where there are N^m subcubes of size $\sim 1/N$. Under f, there are N^m images of size $\sim (1/N)^k$, then the total volume is N^{m-kn}, where we see now the exponent is negative. □

Application 1. *Let $U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$, $N \subseteq \mathbb{R}^n$ a submanifold. Then there are arbitrary sm all $c \in \mathbb{R}^n$ such that $\tilde{f}(x) = f(x) + c$ is transverse to N.*

Proof. In fact, we will show that this is true for almost all c, This question can be addressed locally on N. Take a local parametrization $V \subseteq \mathbb{R}^n$, $g : V \to \mathbb{R}^n$ of N. Then transversality is the samething as the map $U \times V \to \mathbb{R}^n$, given by $(x, y) \mapsto f(x) - g(y)$, has zero as a regular value. But that will be true if we replace f by $\tilde{f}$. Finally, cover N by countably many local charts. □

Definition 13. *Let $U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}^m$ smooth. A fixed point $x = f(x)$ is called nondegenerate if* 1 *is not an eigenvalue of Df_x. (The picture of a degenerate fixed point is that of a "line of fixed points").*

Lemma 6. *If x is nondegenerate, then x is an isolated fixed point.*

Proof. Consider $g(x) = x - f(x)$, then $g(x) = 0$ for fixed point x. And x nondegenerate corresponds to $Dg_x = 1 - Df_x$ is an isomorphism. Now apply inverse mapping theorem. □

Application 2. *There are arbitrarily small c such that $\tilde{f}(x) = f(x) + c$ has only nondegenerate fixed points.*

(Same trick as before.)

Thus in general, when modeling things, we should be able to assume that we only have nondegenerate fixed points (just because of uncertainty in real life), and this eventually became a basic assumption in dynamical systems. (Is this really all that important? Well...)

Definition 14. *$U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}$. A critical point $Df_x = 0$ is nondegenerate if the Hessian $D^2 f_x$ is invertible.*

Lemma 7. *If x is nondegenerate, then x is isolated.*

Proof. Consider $g(x) = Df_x$. x is critial iff $g(x) = 0$, and x is nondegenerate iff Dg is onto. Now apply inverse function theorem. □

Application 3. *Given f, we can perturb it to some $\tilde{f} = f(x) + \sum_i c_i x_i$ for small c_i, such that it has only nondegenerate critical values.*

Application 4. *Let $M, N \subseteq \mathbb{R}^n$ be submanifolds, then there are arbitrarily small $c \in \mathbb{R}^n$ such that for $\tilde{M} = M + c$, $\tilde{M} \cap N$ is a submanifold of dimension* $\dim(M) + \dim(N) - n$.

Proof: look at the situation locally, take the difference of the two manifolds, and make zero a regular value.

Application 5 (Bertini's Theorem)**.** *Let $X \subseteq \mathbb{C}^n$ be a smooth algebraic subvariety (a complex submanifold defined algebraically), then for all but finitely many $c \in \mathbb{C}$, $X \cap \{x_1 = c\}$ is a smooth algebraic subvariety.*

Proof. Use Sard to show that this is true for almost all c (so it is non-empty), then use algebraic geometry to show that the set of c is constructible (either finite or everything except finite) (use elimination theory to make the statement "exists a singular value" into polynomial conditions). □

Need to Know Topology of 1-dimensional submanifolds. Namely, if $M \subseteq U$ is a one-dimensional submanifold, then each connected component of M is either homeomorphic to S^1 or to $\mathbb{R}$.

Theorem 5.2 (Brouwer Fixed Point). *Every bounded continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point.*

Proof. Suppose f has no fixed point, then the same is true of all $\tilde{f}$ with $|\tilde{f}(x) - f(x)| < \varepsilon$ for all x and ε small, because $\mathrm{dist}(x, f(x))$ reaches a minimal value somewhere, as it is never zero and goes to infinity when $x \to \infty$. We can therefore assume without loss of generality that f is smooth.

Consider $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by $F(t, x) = x - \alpha(t)f(x) - c$ where c is a small constant, $\alpha(t)$ is a smooth cutoff function that is 1 when $t < 1$ and 0 when $t > 2$. We can choose c such that 0 is a regular value of F. Look at $F^{-1}(0)$: for $t \gg 0$, $\{x = c\}$; for $t < 0$, $\{x = f(x) + c\}$. If f has no fixed point, then for small enough c, this has no solution. But $F^{-1}(0)$ is a 1-dimensional manifold and is bounded in the $\mathbb{R}^n$-direction ($F(t, x) = 0$ iff $x = \alpha(t)f(x) + c$, if $\|f(x)\| \le C$, then $\|x\| \le C + \|c\|$), yet $x \to \infty$ direction is also impossible, so $F^{-1}(0)$ has nowhere to go. □

Problem 5. *$f : \mathbb{R}^n \to \mathbb{R}$ proper smooth funtion. Assume that all $y \in [a, b]$ are regular values. Show that there is a diffeomorphism of $\mathbb{R}^n$ which takes $f^{-1}(a)$ to $f^{-1}(b)$ (of course those are compact sets because of properness).*

Hint: By constructing a suitable flow that flows through all the intermediate sets. (Outside of a large compact subset make it nothing / identity). All the intermediate values also need to be proper; this is important.

6 Feb 22

Recall Brouwer fixed point theorem. Remark: it is in fact sufficient if there are constants $C < 1$ and D such that $\|f(x)\| \le C\|x\| + D$. This should be thought of as an "topological" version of the Banach fixed point theorem.

Corollary 5. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map of the form $f(x) = A(x) + g(x)$, $A \in \mathrm{GL}(n, \mathbb{R})$, g is bounded, then f is onto.*

Proof. By passing from f to $A^{-1}(f)$ can assume $A = \mathrm{Id}$. Then apply Brouwer fixed point theorem to $-f(x) + x + c = g(x) + c$ where c is an constant. Then there exists some $x = -g(x) + c \implies f(x) = c$. □

Definition 15. *$U, V \subset \mathbb{R}^m$ be open subsets, with v connected, and $f : U \to V$ a smooth proper map. The mod 2 degrees of f is then $\deg_{\mathbb{Z}/2}(f) = |f^{-1}(y)| \in \mathbb{Z}/2\mathbb{Z}$ for any y a regular value.*

Theorem 6.1. $\deg_{\mathbb{Z}/2}(f)$ *is well-defined.*

Proof. Suppose $C \subseteq V$ is a one-dimensional submanifold, the image of an embeding $R \hookrightarrow V$. Suppose f is transverse to C, and consider $f^{-1}(C) \xrightarrow{f} C$. Each connected component of $f^{-1}(C)$ is either a circle or a copy of $\mathbb{R}$.

If we consider a component of $f^{-1}(C)$ which is a copy of $\mathbb{R}$ the number of preimages of a regular point inside that component is either always even or always odd. (Use the intermediate value theorem)

If we consider a component of $f^{-1}(C)$ which is a circle, then corresponding number is always. Outcome: $|f^{-1}(y)|$ mod 2 is the same f_a for all regular values.

To prove the theorem, given the regular values y_0, y_1, one has to find a C which poasses sufficiently closed to y_0 and y_1 and which is transverse to f. In fact, one can take C to be a straight line. □

Theorem 6.2. *$U, V \subset \mathbb{R}^m$ open, V connected , take a proper smooth map f then for $R \times U \to \mathbb{R} \times V$ given by $(t, x) \mapsto (t, F_t(x))$, then $\deg_{\mathbb{Z}_2}(f_t : U \to V)$ is independent of t.*

Remark 5. *Using this and a smoothing process, one can extend action of degree to any continuous map.*

Application 6. *$p : \mathbb{C} \to \mathbb{C}$ complex polynomial of degree d, then $\deg(p) = d$ by deformation to $p(z) = z^d$. This implies the fundamental theorem of algebra if d is odd.*

Application 7. *Is the fundamental theorem of algebra true for quaternions? (Answer: yes.) Consider $p : \mathbb{H} \to \mathbb{H}$ where $p(z) = za_1za_2 \dots za_d +$ lower order terms then p has a root. (Eilenberg-Niven 1940s)*

Suppose d is odd. Recall that $\mathbb{H} = \{A = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}\} \subseteq \mathrm{Mat}_{2\times 2}(\mathbb{C})$. $\det(A) = |a|^2 + |b|^2$ is a multiplicative norm. This implies that p is proper.

To compute the degree, we can deform to $p(z) = z^d$. Then $A = \begin{pmatrix} \zeta & 0 \\ 0 & \overline{\zeta} \end{pmatrix}$, then $|\zeta| = 1$, $\zeta \neq \pm 1$, then it has exactly d preimages $\begin{pmatrix} \zeta^{1/d} & 0 \\ 0 & \overline{\zeta}^{1/d} \end{pmatrix}$. Remains to show that some A is a regular value. This follows from e.g. directly writing out the Jacobian at $A = i$.

Definition 16. *$U, V \subset \mathbb{R}^m$ open, V connected, $f : U \to V$ proper smooth, the degree of f is $\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sgn}(\det(Df_x))$ for y a regular value.*

Theorem 6.3. $\deg(f)$ *is well-defined.*

Theorem 6.4. *If $R \times U \to R \times V$ $(t, x) \mapsto (t, f_t(x))$ proper, then $\deg(f_t)$ is the same for all t.*

Application 8. *$p : \mathbb{C} \to \mathbb{C}$ polynomial of degree d, then $\deg(p) = d \in \mathbb{Z}$. This implies the fundamental theorem of algebra for all d.*

Why? For any p, the derivative is complex multiplication with $p'(z)$ and $\det D_{p_z} = \det \begin{pmatrix} \mathrm{re}(z) & -\mathrm{im}(z) \\ \mathrm{im}(z) & \mathrm{re}(z) \end{pmatrix} = |z|^2 \geq 0$.

Problem 6. *Prove that $p : \mathbb{H} \to \mathbb{H}$, $p(A) = Aa_1Aa_2 \dots Aa_d +$ lower terms has degree $d \in \mathbb{Z}$. (Due to invariance, you don't have to prove this for a general polynomial.)*

Definition 17. *An orientation of a finite-dimensional vector space is an equivalent class of bases $\{v_1, \dots, v_n\}$ and $\{v'_1, \dots, v'_n\}$ are equivalent if $v'_i = \sum_j A_{ij} v_j$ and $\det(A) > 0$.*

Note: an orientation of the space $V = 0$ is defined as an element of $\{\pm 1\}$.

An orientation of V and W determines uniquely one of $V \oplus W$. An orientation of V and $W \subset V$ determines one of V/W.

More generally, given a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$, an orientation of any two determines one for the third.

Definition 18. *An orientation of a submanifold $M \subseteq U \subseteq \mathbb{R}^m$ is an orientation of each target space $TM_x \subseteq \mathbb{R}^m$, locally constant in x.*

Example 10. *The Mobius band does not admit an orientation.*

Lemma 8. *$f : U \to \mathbb{R}^n$, y a regular value, then $f^{-1}(y)$ has a natural orientation.*

If $x \in f^{-1}(y)$, we have $0 \to TM_x \to \mathbb{R}^m \xrightarrow{Df_x} \mathbb{R}^n \to 0$, where the second and third have natural orientations. The resulting orientation of TM_x is locally constant is x.

In particular, you can't have some $\mathbb{R}^3 \to \mathbb{R}$ such that the preimage of a point is a Mobius band.

If $m = n$, the orientation of $f^{-1}(y)$ just assigns for each x the numbrer $\mathrm{sgn}(Df_x) \in \{\pm 1\}$.

Lemma 9. *$U \xrightarrow{f} \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ oriented submanifold, and f is transverse to N, then $f^{-1}(N)$ inherits an orientation.*

For $x \in f^{-1}(N) = M, y = f(x)$, we have $0 \to TM \to \mathbb{R}^m \xrightarrow{Df_x} \mathbb{R}^n/TN_y \to 0$ by transversality. On the other hand, we have $0 \to TN \to \mathbb{R}^n \to \mathbb{R}^n/TN_y \to 0$.

Thus the argument for integer degree being constant actually makes sense.

Remark 6. *Similarly, if $U \xrightarrow{f} U$ smooth, $\overline{f(U)} \subseteq U$ compact and f has nondegenerated fixed points, we define $L(f) = \sum_{x : f(x) = x} \mathrm{sgn}(\det(1 - Df_x))$, this we define as Lefschetz fixed point number. This is 1 in Brouwer's case, and reasonably good deformation preserves this number.*

7 Feb 24

Funny fact: the Euler characteristic of the Hawaiian earring is not well-defined even though it's compact. Okay, how do we even define it? We can look at cohomology, but there's a differential topological approach to it.

Recall: $K \subseteq U \subseteq \mathbb{R}^m$, K compact, U open, $f : U \to K$ a continuous map. This has a well-defined Lefschetz fixed point number $L(f) \in \mathbb{Z}$.

If f is smooth and its fixed points are nondegenerate, then $L(f) = \sum_{x: f(x)=x} \operatorname{sgn}(\deg(1 - Df_x))$. For a general smooth f, replace f by $\tilde{f}(x) = f(x) + c$ to make the fixed points nondegenerate. (Note: $\tilde{f}$ still lands in a compact subset of U). For continuous maps, C^0-approximate by a smooth function.

This is well-defined and homotopy invariant.

Theorem 7.1. *$K \subset U \subset \mathbb{R}^n$, $f : [0,1] \times U \to K$ continuous, then $L(f(0, \bullet)) = L(f(1, \bullet))$.*

Theorem 7.2. *Let $K \subset U$, $\tilde{K} \subset \tilde{U}$ be two different compact inclusions into open in $\mathbb{R}^m$ and $\mathbb{R}^{\tilde{m}}$. Let $f : U \to \tilde{K}, \tilde{f} : \tilde{U} \to K$, then $L(f\tilde{f}) = L(\tilde{f}f)$.*

Lemma 10. *Fix matrices A, B, then $\det(1 - AB) = \det(1 - BA)$.*

Proof. $\det(1 - AB)\det(1 - BA) = \det(A - ABA) = \det(A)\det(1 - BA)$. □

Lemma 11. *$A \in \mathrm{Mat}_{m \times \tilde{m}}(\mathbb{R}), B \in \mathrm{Mat}_{\tilde{m} \times m}(\mathbb{R})$, then $\det(1 - AB) = \det(1 - BA)$.*

Proof. Eigenvectors for nonzero eigenvalues of AB and BA correspond to each other. (On the subspace spanned by the eigenvectors, AB and BA are conjugate.) □

Proof of the theorem. Add constants to f and $\tilde{f}$ to make the fixed points of $f\tilde{f}$, $\tilde{f}f$ nondegenerate. Note that $\tilde{f}f(x) = x \Leftrightarrow f\tilde{f}(y) = y$ where $y = f(x)$, and check local contribution by lemma and we are done. Why can we perturb them to make both nondegenerate? Either by a topological argument (that nondegeneracy is an open condition), or by a measure-theory argument (by Fubini's theorem). □

This allows us to see that Lefschetz number behaves like a trace in some sense.

Definition 19. *A compact subset $K \subset \mathbb{R}^m$ is a neighborhood retract if there is an open U containing K, and a continuous $r : U \to K$, $r|_K = id$.*

Recall that a subset is a submanifold iff it is a smooth neighborhood retract. On the other hand, the Hawaiian earring is not a smooth retract.

Definition 20. *Let $r : U \to K$ be a neighborhood retract, then we define the Euler characteristic of K as $\chi(K) = L(r)$.*

Lemma 12. *This is independent of the choice of U and r.*

Proof. Suppose $r : U \to K, \tilde{r} : \tilde{U} \to K$ are two retractions, then $L(r) = L(\tilde{r}r) = L(r\tilde{r}) = L(\tilde{r})$. □

Lemma 13. *$r : U \to K$ retraction, $\tilde{r} : \tilde{U} \to \tilde{K}$ retraction, where they may live in different $\mathbb{R}^m$ and $\mathbb{R}^{\tilde{m}}$, and we have a homeomorphism $f : K \to \tilde{K}$, then $\chi(K) = \chi(\tilde{K})$.*

Proof. $\chi(K) = L(r) = L(f^{-1}fr) = L(f^{-1}\tilde{r}fr) = L(frf^{-1}\tilde{r}) = L(ff^{-1}\tilde{r}) = L(\tilde{r})$ by the theorem we proved above. □

Lemma 14. *$r, \tilde{r}$ same as before. Assume that there are continuous maps $f : K \to \tilde{K}, \tilde{f} : \tilde{K} \to K$ such that $f\tilde{f}, \tilde{f}f$ are homotopic to the identity, then $\chi(K) = \chi(\tilde{K})$. In other words, it is defined up to homotopy equivalence.*

Example 11. $\chi(S^1) = 0$.

Proof. Let $r(x) = x/|x|$ and $f(x) = \zeta x/|x|$, $\zeta \in S^1 \backslash \{1\}$. Then $L(r) = L(f)$ by homotopy equivalence, but f is fixed point free. □

Problem 7. *Compute the Euler characteristic of S^n.*

Now let's consider the infinite dimensional case. $U \subset H$ open subset of a separable Banach space, $V \subset K$ be the same and also connected. Let $f : U \to V$ be a proper smooth Fredholm map, such that $\mathrm{idx}(Df_x) = 0$ for all x. Pick a regular value y, and define $\deg_{\mathbb{Z}/2}(f) = |f^{-1}(y)| \in \mathbb{Z}/2$. The regular value exists by Sard-Smale. This is independent of y and is called the mod 2 Leray-Schauder degree.

Example 12. *$f : H \to H, f(x) = Ax + g(Kx)$, A invertible, g bounded, K compact operator. Then f is onto because $\deg_{\mathbb{Z}/2} f = 1$ (because we can use homotopy equivalence to kill the second part). This shows that the differential equation always has a solution.*

What about orientation and the $\mathbb{Z}$-valued degree?

Lemma 15. *For any Fredholm operator $F : K \to K$, define $\delta(F)$ to be the set of orientations of $\ker(F) \oplus (K/\operatorname{im}(F))^\vee$. Then, there are narutal identifications $\delta(F) = \delta(\tilde{F})$ for any $\tilde{F}$ sufficiently close to F.*

More rigorously, there is a canonical double covering δ : (all F together with a point in $\delta(F)$) $\to$ (space of all Fredholm operator F). Unfortunately, this is a nontrivial $\mathbb{Z}/2$-bundle, and thus it is not possible to choose orientation for Fredholm operators in a consistent way. Topology of the latter space, which we denote by $\mathcal{F}$: it has $\mathbb{Z}$ connected components, indexed by the index, and each connected component has the fundamental group of $\mathbb{Z}/2$ classified by δ. In fact, $\mathcal{F}$ is weakly homotopy equivalent to $\mathbb{Z} \times BO$ (this is a theorem of Atiyah and Jänich).

Corollary 6. *If $F : H \to K$ is Fredholm, the preimage of a regular value is not necessarily oriented.*

Corollary 7. *$\mathbb{Z}$-valued degree can't be defined for Fredholm maps of index 0.*

However, these difficulties disappear if we restrict to Fredholm operator of the form $A + K$ for A fixed invertible, K compact. For example, for maps $f(x) = Ax + g(Kx)$, g bounded, we can define $\mathbb{Z}$-degree which is known as the Leray-Schauder degree.

Theorem 7.3 (Sard-Smale). *H, K separable Banach spaces, $U \subseteq H$ open, $F : U \to K$ smooth Fredholm, then the set of singular values of f is thin.*

Sketch of proof. Use separability to convert this to a local problem. After coordinate change, assume we have a splitting $H = H' \oplus H''$, $K = K' \oplus K''$, H'', K'' finite dimensional, where $f(x', x'') = (Ax', g(x', x''))$, where A invertible. The set of singular values of f are then $\coprod_{x'} = \{Ax'\} \times \{\text{singular values of } g \text{ with } x' \text{ fixed}\}$. Therefore regular values are dense. For similar reasons, f is locally closed. Thus critical values are countable union of closed subsets. □

8 Feb 29

Application 9. *$U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$ smooth map. If $n \geq 2m$, then we can perturb f to get an immersion.*

(Matrices that are not injective is actually not a submanifold, so we can't just use transversality.)

In fact, we are looking at perturbations $f(x) + Ax$, where A is a constant matrix. This will be an immersion if 0 is a regular value of $U \times (\mathbb{R}^m - \{0\}) \xrightarrow{F_A} \mathbb{R}^m$ where $(x, \zeta) \mapsto Df_x\zeta + A\zeta$.

(because of the dimension of the assumption and the homogeneity of F_A in ζ, 0 is a regular value iff $F_A^{-1}(0)$ is a submanifold of dim ≤ 0 iff $F_A^{-1}(0) = \emptyset$ (when $2m = n$, we can't get a submanifold of dim 0: consider homogeneity in ζ), which is iff $D_x f + A$ being always injective.)

Take "all choices at once": $U \times (\mathbb{R}^m - \{0\}) \times \mathrm{Mat}(n, m) \xrightarrow{F} \mathbb{R}^n$ given by $(x, \zeta, A) \mapsto Df_x(\zeta) + A\zeta$. Consider (where we fix x and ζ and alter α): $DF_{(x,\zeta,A)}(\bullet, \bullet, \alpha) = \alpha\zeta$. Since $\zeta \neq 0$, $\alpha\zeta$ can be any vector, so F is a submersion, hence 0 is a regular value of F, and take $M = F^{-1}(0)$.

Now 0 is a regular value of F_A iff A is a regular value of the projection $M \to \mathrm{Mat}(m, n)$, which follows from a direct linear algebraic check. Such A is almost everywhere by Sard. (This uses Sard's theorem for a map from a submanifold, but it's unproblematic to work in local coordinates of M.)

Application 10. *$U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$ with $2m < n$, then we can perturb f to an injective immersion.*

Consider the same perturbation $f(x) + Ax$. We look at $U \times U - \{\text{diagonal}\} \xrightarrow{F_A} \mathbb{R}^n$ given by $(x, y) \mapsto f(x) + A(x) - f(y) - A(y)$. We want to show that for a generic choice of A, 0 is a regular value. Consider the universal problem $(U \times U - \{\text{diagnoal}\}) \times \mathrm{Mat}(m, n) \xrightarrow{F} \mathbb{R}^n$ where $(x, y, A) \mapsto f(x) - f(y) + A(x - y)$. Again $DF_{(x,y,A)}(\bullet, \bullet, \alpha) = \alpha(x - y)$ is always onto. Take $M = F^{-1}(0)$, then 0 is a regular value of F_A iff A is a regular value of $M \to \mathrm{Mat}(m, n)$.

Remark 7. *If $m = 2n$, one can perturb f to be an immersion with only transverse self-intersection. (Same proof applies.) In fact one can prove that no triple intersection can occur.*

Remark 8. *Alternatively, we could consider the "universal" perturbation, i.e. consider $(U \times U - \{\text{diagonal}\}) \times (U \to \mathbb{R}^n) \xrightarrow{F} \mathbb{R}^n$ given by $(x, y, f) \mapsto f(x) - f(y)$. Then $DF_{x,y,f}(\bullet, \bullet, \varphi) = \varphi(x) - \varphi(y)$ is certainly onto. In principle, $M = F^{-1}(0)$ "should" be a submanifold of codimension n, and one could argue by projection $M \to (U \to \mathbb{R}^n)$, which one can see is automatically Fredholm. Technical point: this will work if we choose a suitable saparable Banach space of functions. (But $(U \to \mathbb{R}^n)$ is not in general a Banach space, so one need to choose a Banach subspace on which one requires derivatives to decay fast enough.) Once we have this, Sard-Smale applies.*

Remark 9. *This strategy works for algebraic geometry as well.*

Problem 8. *Let $f : \mathbb{R} \to \mathbb{R}^3$ be an embedding. Show for a generic line projection $P : \mathbb{R}^3 \to \mathbb{R}^2$, $P \circ f$ is an immersion.*

Now let's talk about partition of unity.

Definition 21. *Let $U \subset \mathbb{R}^m$ be an open subset, $U = \bigcup_{\alpha \in A} U_\alpha$ an open cover. A subordinate partition of unity is a collection of smooth functions $(\psi_\alpha)_{\alpha \in A}$ such that*

1. *$\psi_\alpha \geq 0$.*
2. *$\mathrm{supp}(\psi_\alpha) = \overline{\{x : \psi_\alpha(x) \neq 0\}}$ is contained in U_α.*
3. *For each $x \in U$, there is an open subset $V \subset U$ containing x such that $\{\alpha \in A : \psi_\alpha(y) \neq 0 \text{ for some} y \in V\}$ is finite.*
4. *$\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in U$.*

Why don't we just say at each point finitely many of them don't vanish? Because we need functions constructed from partition of unity (e.g. $\sum_{\alpha \in A} \psi_\alpha(x)$) to be smooth.

Theorem 8.1. *Subordinate partitions of unity always exist.*

Application 11. *$U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}$ continuous, $g : U \to (0, \infty)$ continuous. Then there is a smooth function $\tilde{f} : U \to \mathbb{R}$ such that $|\tilde{f}(x) - f(x)| \leq g(x)$ everywhere.*

Proof. Take an open cover $U = \bigcup_{\alpha \in A} U_\alpha$ such that each U_α is an open ball and $\overline{U_\alpha} \subset U$. There are constants ε_α such that $g(x) \geq \varepsilon_\alpha$ for all $x \in \overline{U_\alpha}$. Can find smooth functions $\tilde{f}_\alpha : U_\alpha \to \mathbb{R}$ such that $|\tilde{f}_\alpha(x) - f(x)| \leq \varepsilon_\alpha$ for all $x \in U_\alpha$, then this local problem is solved by convoluting with a smooth function. Set $\tilde{f}(x) = \sum_\alpha \psi_\alpha \tilde{f}_\alpha$ for some partition of unity, then this is defined and is smooth at all of U. Then $|\tilde{f}(x) - f(x)| \leq \sum_\alpha \psi_\alpha |\tilde{f}_\alpha - f| \leq \sum_\alpha \psi_\alpha g = g(x)$. □

Corollary 8. *$U \subset \mathbb{R}^m$ open, then there is a smooth and proper, bounded below function $f : U \to \mathbb{R}$.*

By considering the level set, we now know that U is homotopy equivalent to a countable CW complex. (This is not immediately obvious–consider the complement of the Cantor set.)

More precisely: $x \mapsto (x, f(x)) : U \to \mathbb{R}^{m+1}$ embeds U as a submanifold of $\mathbb{R}^{m=1}$.

Note: This is not true in complex geometry (because you don't have enough functions). c.f. Stiefel manifolds.

Proof. Wlog $U \neq \mathbb{R}^m$, take $g(x) = \dfrac{1}{\text{dist}(x, \mathbb{R}^m - U)} + |x|^2 : U \to (0, \infty)$ and approximate it uniformly by a smooth function. (The $|x|^2$ part is to control the behavior at ∞ for properness.) □

Corollary 9. *$U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ open, $f : U \to V$ continuous. Then f is homotopic to a smooth map.*

Proof. Let $f(x) = (f_1(x), \ldots, f_m(x)), g(x) = \text{dist}(f(x), \mathbb{R}^n - V)$. Find smooth functions $\tilde{f}(x) = (\tilde{f}_1(x), \ldots, \tilde{f}_m(x))$ such that $|\tilde{f}_i(x) - f_i(x)| \leq \dfrac{1}{2n} g(x)$, then $\|\tilde{f}(x) - f(x)\| \leq \dfrac{1}{2}\text{dist}(f(x), \mathbb{R}^n - V)$, Hence $H_t(x) = (1-t)f(x) + t\tilde{f}(x) \in V$ for all $x \in U, t \in I$. □

9 March 02

Application 12. *U an open subset, $M \subset U$ a submanifold, then every smooth function on M can be extended to a smooth function on U.*

Proof. Choose open cover $U = U_a$ such that each U_a is either contained in a local coordinate chart for a submanifold $\varphi_a : U_a \to (0,1)^m \subset \mathbb{R}^m, \varphi_a(M) = (0,1)^k \times \{0\}^{m-k}$, or disjoint from M. Fix $f : M \to \mathbb{R}$ smooth. On each U_α there is a $g_a : U_a \to R$ with $g_a|_{U_a \cap M} = f|_{U_a \cap M}$ (this is the definition for f to be smooth). Set $g(x) = \sum_a \psi_a(x) g_a(x)$, for $x \in M$, easy to verify that $g(x) = f(x)$. □

Corollary 10 (Isotopy Theorem)**.** *$U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ open subsets. Take a map $\mathbb{R} \times U \to V$ given by $(t, x) \mapsto f_t(x)$, which is an embedding for each t, and independent of t outside a compact subset of U. Then there is a family of diffeomorphisms $\psi_t : V \to V$, $\psi_0 = id$, $\psi_t = id$ outside a compact subset such that $f_t = \psi_t \circ f_0$.*

Proof. We want to construct a time-dependent vector field X_t on V such that $X_t(f_t(x)) = \partial f_t / \partial_t(x)$. This will yield the desired family of diffeomorphisms provided that $X_t = 0$ outside a compact subset. But this is an extension problem for a function as in the following diagram:

$$\begin{array}{ccc} \{(t, \partial f_t/\partial t(x))\} & \hookrightarrow & \mathbb{R} \times V \\ & \searrow \quad \mathbb{R}^n \quad \swarrow & \end{array}$$

and we can use the extension theorem for functions (checking from the proof that support condition can be preserved). □

A somewhat more rigorous statement is as follows. (Not sure if we really care.)

Corollary 11. *$\mathbb{R} \times U \to \mathbb{R} \times V$ ($(t, x) \mapsto (t, f_t(x))$) which is an embedding. Moreover, for any compact $I \subset \mathbb{R}$, there is a compact $K \subset U$ such that $\partial f / \partial t = 0$ for $t \in I, x \notin K$, then there is a diffeomorphism $(t, x) \mapsto (t, \psi_t(x))$ such that $\psi_0(x) = 0x, \psi_t(f_0(x)) = f_t(x)$, and for each compact $I \subset \mathbb{R}$ there is a compact $L \subset V$ such that $\partial \psi / \partial t = 0$ for $t \in I, y \notin L$.*

Problem 9. *$U \subset \mathbb{R}^m$ ($m \geq 2$) open connected, $x_1, \ldots, x_r \in \mathcal{U}$ pairwise different points, $y_1, \ldots, y_r \in U$ pairwise different, then there is a difeomorphism $f : U \to U, f(x_i) = y_i$. Note: this would be false for $m = 1$.*

Now let's prove the existence of partition of unity.

Proof of Partition of Unity. Take an open cover $U = U_\alpha$, we can find a cover V_β such that each $\overline{V_\beta}$ is contained in some U_α, $\bigcup V_\beta$ is locally finite (i.e. each point has a neighborhood that intersect only finitely many V_β), this is given by the fact that $\mathbb{R}^n$ is paracompact.

Each V_β is a ball $V_\beta = B_{r_\beta}(x_\beta)$, the balls $B_{r_\beta/3}(x_\beta)$ still cover U. Now for each β, choose $\rho_\beta : U \to [0, \infty)$ smooth, such that $\rho_\beta > 0$ in $B_{r_\beta/3}(x_\beta), \rho_\beta = 0$ outside $B_{r_\beta}(x_\beta)$. Then $\rho = \sum_\beta \rho_\beta$ is smooth and everywhere positive on U, define $\sigma_b\eta = \rho_\beta/\rho$ this is a partition of unity for V_β. To get the necessary partition, devide $B = \coprod_{a\in A} B_a$ such that if $\beta \in B_a, \overline{V_\beta} \subset U_a$, then set $\psi_a = \sum_{\beta\in B_a} \sigma_\beta$. □

Now let's talk about manifolds.

Definition 22. *A topological manifold is a Hausdorff space with a countable denst subset which is locally homeomorphic to $\mathbb{R}^n$.*

A differentiable atlas on a topological manifold M is a collection $f_\alpha : V_\alpha \to U_\alpha$, where U_α open cover M, $V_\alpha \in \mathbb{R}^m$ open, each is a homeomorphism, and $f_{\beta^{-1}} f_\alpha : f_\alpha^{-1}(U_\beta) \to f_\beta^{-1}(U_\alpha)$ is a diffeomorphism for all (α, β). Two atlases f, g are equivalent if the identity map $f_\alpha^{-1} g_\beta$ and its inverse $g_\beta^{-1} f_\alpha$ are both smooth. (As an counterexample, consider two atlas on $\mathbb{R}$ given by $x \mapsto x$ and $x \mapsto x^3$.) Equivalently, their disjoint union is again an atlas. This is an equivalence relation.

Remark 10. *Any topological manifold can have non-equivalent atlases, or no atlas at all.*

(Charles: note that when dimension is less than 4, any topological manifold has a unique differential structure up to diffeomorphism, so the distinction only starts to appear in higher dimensions.)

Under this definition, the following concepts are canonically defined:

- Diffeomorphism of manifolds
- immersions
- submersions
- Embeddings
- Submanifolds

Note that a submanifold is itself an abstract manifold.

Example 13. $\mathbb{RP}^n$ *= set of lines through origin in* $\mathbb{R}^{n+1}$*. Fix a line* $L \subset \mathbb{R}^{n+1}$, $\mathbb{R}^{n+1} = L \oplus H$ *then nearby lines are graphs* $L_A = \{x + Ax, x \in L\}$ *where* $A : L \to H$ *is a linear map. This gives a chart* $(\mathrm{Hom}(L, H) \cong \mathbb{R}^n) \to \mathbb{RP}^n$ *Have to check coordinate changes: in fact, the transition function between the two obvious charts is given by:* $z \mapsto 1/z$.

Example 14. $\mathbb{CP}^n$ *= set of complex lines through the origin in* $\mathbb{C}^{n+1}$*. This is a* $2n$*-dim real manifold. Chart conversion:* $z \mapsto 1/z$.

Example 15. $\mathbb{HP}^n$ *= set of all subspaces* $L \subset \mathbb{H}^{n+1}$ *such that* $\dim_\mathbb{R} L = 4$, L *is invariant left multiplication with* $\mathbb{H}$*. Chart conversion:* $\mathbb{H}(z, 1) \mapsto \mathbb{H}(1, z)$.

Example 16. $\mathbb{O}$ *octonions. We can define* $\mathbb{OP}^1 \cong S^8$ *by taking two charts* $\mathbb{O}$ *and identifying th* $z \mapsto z^{-1}$*. Define a "line" in* $\mathbb{O} \oplus \mathbb{O}$ *to be a subspace consisting of* $\{(\alpha z, \alpha) : \alpha \in \mathbb{O}\}$ *or* $\{(\alpha, \alpha z) : \alpha \in \mathbb{O}\}$ *(for some* $z \in \mathbb{O}$ *fixed) Note:* $y = x^{-1}(xy)$.

You can make $\mathbb{OP}^2$ but it doesn't parametrize $\mathbb{O}^3$ so isn't really helpful. So why do we introduce these? Consider the correspondence {line+unit length point on that line}, and (unit point) $\to$ (line throught point). Then we get maps $S^n \to \mathbb{RP}^n, S^{2n+1} \to \mathbb{CP}^n, S^{4n+3} \to \mathbb{HP}^n, S^{15} \to \mathbb{OP}^1$. The fibers are correspondingly S^0, S^1, S^3, S^7. Now notice the next map $S^{23} \to \ldots$ doesn't exist! These are called Hopf fibrations and they have special roles in algebraic topology.

10 March 07

Definition 23. *X Hausdorff space, Γ (discrete) group acting on X. We say that the action is properly discontinuously if $\forall x_1, x_2 \in X$, there are neighborhoods $x_1 \in U_1 \subset X, x_2 \in U_2 \subset X$ such that $\{\gamma \in \Gamma \mid \gamma(U_1) \cap U_2 \notin \emptyset\}$ is a finite set.*

It's the same thing as saying that $\Gamma \times X \to X$ is a proper map.

Lemma 16. *If Γ acts properly discontinuously, then $\Gamma \backslash X$ is itself Hausdorff.*

Example 17. *$X = S^1$, $\mathbb{Z}$ acting on X by $(k, e^{it}) \mapsto e^{it+k\theta}$, $\theta \notin 2\pi\mathbb{Q} \implies X/\mathbb{Z}$ is not Hausdorff.*

Proof. Take x_1, x_2 and neighborhoods U_1, U_2 as before. Take $F = \{\gamma \in \Gamma : \gamma(U_1) \cap U_2 = \emptyset\}$. Assuming that x_1 is not in the orbit of x_2, we can shrink U_1, U_2 so that for a single $\gamma \in \Gamma$, $\gamma(U_1) \cap U_2 = \emptyset$. Repeat that for all $\gamma \in F$. □

Lemma 17. *If Γ acts properly discontinuously and freely, then $X \to \Gamma \backslash X$ is a local homeomorphism (in fact a covering map).*

Application 13. *If M is a topological manifold and Γ acts freely and properly discontinuously, $\Gamma \backslash M$ i a topological manifold.*

Application 14. *If H is a smooth manifold, and Γ acts freely and properly discontinuously by diffeomorphisms, then $\Gamma \backslash M$ is naturally a smooth manifold.*

Lemma 18. *Let G be a (Hausdorff) topological group, and $\Gamma \subset G$ a discrete subgroup, Then the left action of Γ on G is free and properly continuous.*

Proof. (Sketch) Take $g_1, g_2 \in G$, fix a neighborhood of $g_1 g_2^{-1}$ containing at most one point of Γ, then translate it back in the neighborhoods of g_1 and g_2. □

Lemma 19. *Let G, Γ be as before, and K a compact subgroup. Then Γ acts properly discontinuously on G/K.*

Example 18. *$G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}_/ \pm I$. $K = \mathrm{PSO}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R})/\pm I$. Note that G acts on the upper half plane $H = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ by $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} : z \mapsto (az + c)/(bz + d)$. The stabilizer of $z = i$ is K.*

So we get a continuous map $G/K \to H$ where $A \mapsto Ai$. This is bijective and has a continuous inverse $z \mapsto \begin{pmatrix} \mathrm{im}(z) & 0 \\ \mathrm{re}(z) & 1 \end{pmatrix}$, hence is a homeomorphism. Given a discrete subgroup $\Gamma \subset G$, we get a Hausdorff space $\Gamma \backslash G/K = \Gamma \backslash H$, which is a two-dimensional differentiable manifold if Γ acts freely on $G/K = H$.

Conjugacy classes in $\mathrm{PSL}_2(\mathbb{R}) = G$:

1. $A \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $0 < \lambda < 1$ (hyperbolic) acts as $z \mapsto \lambda^2 z$ (free)

2. $A \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $c \neq 0$ (parabolic) act as $z \mapsto z + c$ (free)

3. $A \sim \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, $\theta \notin \pi\mathbb{Z}$ (elliptic), act with a single fixed point.

Note: A elliptic iff $\mathrm{tr}(A)^2 < 4$.

Consider for instance (N positive integer) $\Gamma(N) = \{A \in \mathrm{PSL}_2(\mathbb{Z}) : A \equiv \pm I \pmod N\}$, Then $\mathrm{tr}(A) \equiv \pm 2 \pmod N$, so if $N > 3$, $\mathrm{tr}(A) \notin \{0, \pm 1\}$. In that case, $\Gamma(N)$ acts freely, so $\Gamma(N) \backslash G/K = \Gamma(N) \backslash H = X(N)$, which is a smooth Riemannian surface (modular curve).

What if $\Gamma \subset G = \mathrm{PSL}_2(\mathbb{R})$ discrete, but not acting freely on H? (e.g. $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ has points with stabilizers $\mathbb{Z}/2$ and $\mathbb{Z}/3$.) Given any $z \in H$ with stabilizer Γ_z, the quotient $\Gamma \backslash H$ looks locally like $\Gamma_z \backslash H$ near x. From an analysis of $z = i$, we know that Γ_z is a finite cyclic group. Hence, the quotient looks like

$\mathbb{C}/(k\text{th roots of unity}) \cong \mathbb{C}$ ($z \mapsto z^k$). Hence $\Gamma\backslash H$ is still a topological surface. In fact, one can make $\Gamma\backslash H$ into a Riemannian surface using removal of singularities. So $\mathrm{PSL}_2(\mathbb{Z})\backslash H \cong \mathbb{C}$ (call the map from left to right j functions), but $\mathbb{C}$ carries two special points (images of the orbits with $\mathbb{Z}/2$ and $\mathbb{Z}/3$ isotropy). In fact, $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}/2 \times \mathbb{Z}/3$.

Example 19. *Let $\mathrm{O}(n,1)$ be the group of linear automorphisms of $\mathbb{R}^{n+1}$ which preserve the quadratic form $P(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2$. In fact, they preserve $\{x : P(x) = -1\} = \{x : x_{n+1}^2 = x_1^2 + \ldots + x_n^2 + 1\}$. (This is a hyperboloid with two components—conveniently named "future" and "past.")*

Let $\mathrm{SO}(n,1) = \{A \in \mathrm{O}(n,1) : \det(A) > 0, A$ preserves the connected components of the hyperboloid$\}$. This acts on $H_n = \{x : P(x) = -1, x_{n+1} > 0\}$, and the stabilizer of the point $x = (0,\ldots,0,1)$ is the subgroup $\mathrm{SO}(n) \subset \mathrm{SO}(n,1)$ Hence we get for $G = SO(n,1), K = SO(n)$ that $G/K \cong H_n$. Therefore if $\Gamma \in G$ is a discrete subgroup acting freely on $G/K = H_n$, we get a quotient n-manifold $M = \Gamma\backslash G/K$. These are called hyperbolic n-manifolds.

If $n = 2, SO(2,1) \cong \mathrm{PSL}_2(\mathbb{R})$, and the action of H_2 is the usual one on the upper half plane. If $n = 3$, $SO(3,1) \cong \mathrm{PSL}_2(\mathbb{C})$, which are automorphisms of the Riemann sphere $\mathbb{CP}^1$, thought of as lying at ∞ in the light cone.

Let $K = \mathrm{Q}(\sqrt{d}) \subset \mathbb{C}, d < 0$ be a quadratic number field. The ring of integers $\mathcal{O} = \mathcal{O}(K) \subset \mathbb{C}$ forms a lattice in the complex plane. We can then set $\Gamma = \mathrm{PSL}_2(\mathcal{O})$, which is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. If Γ acts freely on H_3, we get a hyperbolic 3-manifold $M = \Gamma\backslash H = G\backslash G/K$. These are called arithmetic 3-manifolds.

Remark 11. *For $G/K = H_3$, even if Γ does not act freely, $\Gamma\backslash H_3$ is still a topological 3-manifold. Reason: if $\Xi \subset \mathrm{GL}(3,\mathbb{R})_+$ is a finite group, $\mathbb{R}^3/\Xi \cong \mathbb{R}^3$.*

Problem 10. *Is $\{(x,y) \in \mathbb{C}^2 : x^3 = y^5\}$ a topological manifold?*

Problem 11. *Give an example of $\Xi \subset \mathrm{GL}(4,\mathbb{R})_+$ finite, so that $\mathbb{R}^4/\Xi$ is not a 4-manifold (with proof).*

11 March 09

Consider n-tuples $(x_1,\ldots,x_n)$ in $\mathbb{R}$, $x_i \neq x_j$ for $i \neq j$, we declare two such to be equivalent if they differ by an affine automorphism of $\mathbb{R}$ ($x_i' = ax_i + b, a \in \mathbb{R} - \{0\}, b \in \mathbb{R}$). Let $M_{0,n+1}(\mathbb{R})$ be the set of equivalence classes.

$M_{0,3}(\mathbb{R}) = \{\text{point}\}$.

$M_{0,4}(\mathbb{R}) =$three open intervals.

$M_{0,n+1}(\mathbb{R}) = \dfrac{n!}{2}$ open simplices of dimension $n-3$.

There is a distinguished compactification $\overline{M_{0,n+1}}(\mathbb{R})$ optained by letting points run into each other and looking at them under a microscope.

(Picture with "screens" where the individual "points" are points in $M_{0,j}(\mathbb{R}$ for $j \geq 3)$.

This is a description of $M_{0,n+1}(\mathbb{R})$ as a set (and with some imagination as a topological space).

Then we see that $\overline{M_{0,4}}(\mathbb{R}) \cong \mathbb{S}^1$, where we have a nice picture.

In $\overline{M_{0,5}}(\mathbb{R})$, consider the closure of one connected component of $M_{0,5}(\mathbb{R})$. It's not hard to see that it's a pentagon, and realize it's obtained by gluing 12 such pentagons. But it's not S^2! Because the number of pentagons meeting at a point isn't right. But if we compute the Euler characteristic, we see it's $12 - 30 + 15 = -3$. It turns out it is connected sum of five copies of $\mathbb{RP}^2$, and is Hausdorff. It is a smooth manifold, but it's not obviously so.

Remark: there is a complex counterpart $\overline{M}_{0,n+1}(\mathbb{C})$ using points in the plane. They are called Deligne-Mumford spaces.

Remark: $\overline{M}_{0,6}(\mathbb{R})$ is obtained by gluing together 60 copies of the following polytope: a cube cutting three non-sharing-vertex edges off.

The outcome is an Eilenberg-Maclane $K(\pi,1)$, for π infinite.

Let's talk about orientation. M manifold, $y \in M$ point, take local chart $U \subset \mathbb{R}^n \xrightarrow{\psi} V \subset M, \psi(x) = y$, $U' \xrightarrow{\psi'} V'$, $\psi'(x') = y$. Two such charts induce the same local orientation at y if $D_x(\psi')^{-1}\psi$ has positive determinant. This is an equivalence relation, and there are exactly two equivlance classes called local orientations at y. A local orientation at y induces a local orientation at any point sufficiently close to y.

We define $M^\circ = \{(y, \circ) : y \in M, \circ \text{ is a local orientation at} y\}$.

Then M° has a 2-to-1 quotient map to M, and $\mathbb{Z}/2$ thus acts on M°. and M° carries a canonical topology as a covering space, Hence M° is canonically a manifold, and is called the orientation covering of M.

Definition 24. *M is orientable if $M^\circ \to M$ is the trivial covering. An orientation of M is a choice of section $M \to M^\circ$.*

Remark 12. *M° is always orientable and has a canonical orientation.*

Remark 13. *Notion of orientable / unorientable loop in any M is then well-defined (by looking at whether the two sheets get swapped). Then M is orientable if all loops are orientable.*

Remark 14. *If there are no nontrivial double covers (e.g. M connected, $\pi_1(M)$ trivial) then M is orientable.*

What about $S^n \to \mathbb{RP}^n$? If this is the orientation cover then we know $\mathbb{RP}^n$ is unorientable, since the covering is clearly not trivial.

Example 20. *$U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$ smooth. If 0 is regular value, then $M = f^{-1}(0)$ is always oriented.*

To see this, let $x \in \mathbb{R}^{m-n}$. Map it to $y \in M$, then by f to $0 \in \mathbb{R}^n$. ψ is compatible with the orientation iff $(D\psi_x(1, 0, \ldots, 0), \ldots, D\psi_x(0, \ldots, 0, 1), v_1, \ldots, v_n)$ is a positively orientated vector in $\mathbb{R}^m$, where $Df_y(v_1) = (1, 0, \ldots, 0), Df_y(v_2) = (0, 1, 0, \ldots)$, etc.

Example 21. *M, N oriented, $f : M \to N$ has y as regular value, then $f^{-1}(y)$ is oriented.*

Lemma 20. *A double cover $\tilde{M} \xrightarrow{\tau} M$ is isomorphic to the orientation cover if and only if $\tilde{M}$ is oriented and the $\mathbb{Z}/2$ action reverses the orientation.*

Example 22. *$\mathbb{RP}^n$ is not orientable for n even, because the antipodal map on S^n reverse orientations. It is orientable for n odd, since then antipodal map preserves orientations.*

Remark 15. *Suppose $M \subset \mathbb{R}^{n+1}$ is a hypersurface (n-dimensional submanifold), then $M^\circ \cong \{y = x + v, x \in M, v \perp TM_x, \|v\| = f(x)\}$ for a sufficiently small $f : M \to (0, \infty)$.*

Problem 12. *$M \subset \mathbb{R}^{n+1}$ is a hypersurface then it is orientable. (Do not use algebraic topology.)*

12 March 14

Orientation of manifolds.

Application 15. *Let M be a complex (analytic) manifold. Then, as a real manifold, M has a canonical orientation. If we have a complex chart $U \to M$ we get a real chart, and two such charts induce the same local orientation, since for $A \in \mathrm{GL}_n(\mathbb{C})$, $\det_{\mathbb{R}}(A) = |\det_{\mathbb{C}}(A)|^2 > 0$.*

Application 16 (Blowing up a point)**.** *$P = \mathbb{KP}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (in this case $n = 1$). $L = \{(x, \lambda) \in \mathbb{K}^{n+1} \times P : x \in \lambda\}$. Then $L = P \cup (\mathbb{K}^{n+1} \backslash \{0\})$. Hence L is a manifold which is diffeomorphic to $\mathbb{K}^{n+1}$ outside a compact subset. Take any connected manifoldld M^m, $m = \dim(M) = \dim(L)$. Fix a chart $\mathbb{K}^{n+1} = \mathbb{R}^m \xrightarrow{\varphi} M$ and set $Bl_{\mathbb{K}}(M) = (M \backslash \varphi^{-1}(0) \cup L)/\sim$, where the equivalence relation identifies $x \in \mathbb{K}^{n+1} \backslash \{0\} \subset L$ with $\varphi(x) \in M$.*

The diffeomorphism type of $Bl_{\mathbb{K}}(M)$ remains unchanged if we smoothly deform the chart. (Local isotopy argument). If M is not orientable, any chart can be deformed to any other chart, hence the blowup is well defined up to diffeomorphism. The same is true whenever L admits a diffeomorphism which, outside a compact subset, is an orientation-reversing linear map of $\mathbb{K}^{n+1}$. For instance, this is true for $\mathbb{K} = \mathbb{R}$. In all cases, we can make it well-defined by specifying an orientation. In general, blowup is well-defined for oriented manifolds using oriented local charts.

Example 23. *$Bl_{\mathbb{R}}(S^n) = \mathbb{RP}^n$, since the projection $\mathbb{R}^{n+1} \xrightarrow{x_1} \mathbb{R}$ identifies $\mathbb{RP}^{n+1}\backslash\{\text{point}\} \to L$. Now glue back the remaining point that is not covered by the chart.*

Example 24. *Similarly, $Bl_{\mathbb{C}}(S^{2n}) = \mathbb{CP}^n$ and $Bl_{\mathbb{H}}(S^{4n}) = \mathbb{HP}^n$.*

Definition 25. $\mathbb{OP}^2 = Bl_{\mathbb{O}}(S^{16})$.

Example 25. *$Bl_{\mathbb{C}}(\mathbb{CP}^2)$ yields a 4-manifold with intersection form $\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}$ or $\begin{pmatrix} \pm1 & 0 \\ 0 & \mp1 \end{pmatrix}$ depending on the choice of chart. (The second one gives the complex oriented charts.)*

Connected Sum Take M, N connected manifolds, charts $\varphi : \mathbb{R}^m \to M, \psi : \mathbb{R}^m \to N$, then $M\#N = (M\backslash\varphi(0) \cup N\backslash\psi(0))/\sim$ or $(M\backslash\varphi(B^m) \cup N\backslash\psi(B^m))/\sim$ where in the first we identify $\varphi(x)$ with $\psi(x/\|x\|^2)$, and in the second we do $\varphi(x) \sim \psi(x)$ for $\|x\| = 1$.

If one of the two manifolds is not orientable, or admits an orientation-reversing diffeomorphism, then the connected sum is well-defined as an unoriented manifold. If both are oriented, choose φ compatibly oriented, ψ not compatibly oriented, then the connected sum is well-defined as an oriented manifold.

Example 26. *$Bl_{\mathbb{R}}(M^n) = M\#\mathbb{RP}^n$, which is well defined since $\mathbb{RP}^n$ admits an orientation-reversing diffeomorphism (coming from that of S^n by flipping a coordinate).*

Example 27. *Let M be a complex manifold. The complex-analytic blowup is $M\#\overline{\mathbb{CP}^n}$, where the overline indicates we reverse the given (complex) orientation. For instance, the Hirzebruch surface F_1 is $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.*

Remark 16. *Oriented manifolds homotopy equivalent to S^n, fixed $n \geq 5$, up to diffeomorphism, along with connected sum as the additive operation, is a finite abelian group where S^n is the identity, and orientation reverseal is the antipodal map. It is denoted by Θ_n e.g. $\Theta_7 = \mathbb{Z}/28$.*

Degree Let M, N be compact manifolds of dimension n, where M, N are connected. Take $\varphi : M \to N$ smooth, fix $y \in N$ an regular value, then $\deg(\varphi)$ is defined as the number of preimages of y mod 2. This is well-defined (independent of y) and unchanged under deformations of φ.

Proposition 1. *if $M \xrightarrow{\varphi} N \xrightarrow{\psi} O$ are maps of connected n-manifold, then $\deg(\psi\varphi) = \deg(\varphi)\deg(\psi)$.*

Oriented Degree M, N compact oriented connected manifolds. $\varphi : M \to N$ smooth map, $\varphi(x) = y$. y regular point. We say x is positive if $D_x\varphi$ has positive determinant in oriented local charts,negative otherwise. If y is a regular value, $\deg(\varphi) = \sum_{x\in\varphi^{-1}(y)} \pm1 \in \mathbb{Z}$. This has the same properties as before.

Example 28. *An orientation-preserving diffeomorphism $\varphi : M \to M$ and orientation-reversing diffeomorphism $\varphi : M \to M$ has $\deg(\varphi) = 1, \deg(\psi) = -1$. A constant map $M \to M$ has degree 0. (They can't be deformed into each other.)*

Problem 13. *Let M, N be compact connected complex (analytic) manifolds of the same dimension. Let $\varphi : M \to N$ be a complex map. Then either $\varphi(M)$ has no interior points, or $\varphi(M) = N$.*

13 March 28

Basic techniques we have developed so far for the differential world (and how they apply in the algebraic setting):

- Local structure of differential maps. (This doesn't work very well with polynomial maps, but better with formal power series. This was actually what was first developed: formal coordinate changes.)
- Partition of unity: usually works for compact manifolds. (Certainly not working for algebraic setting.)
- Sard's theorem and consequences. (It applies for algebraic setting–say, Bertini's theorem), complex numbers, and (trivially) finite fields.) (Check: Poonen's paper for finite field Bertini theorem).

Sample results:

Proposition 2. *$f : M \to N$ proper injective immersion (define it in local charts), then $f(M) \subset N$ is a submanifold.*

Proposition 3. *$f : M \to N$, $y \in N$ regular value (defined in local chart), then $f^{-1}(y) \subset M$ is a submanifold.*

Proposition 4. *$f : M \to N, P \subset N$ a submanifold such that f is transverse to P (defined locally), then $f^{-1}(P)$ is a submanifold.*

Theorem 13.1 (Ehresmann)**.** *$f : M \to N$ proper, $y \in N$ regular, then $\exists$ open subset $y \in V \subset N$ and a diffeomorphism $f^{-1}(V) \xrightarrow{\Psi} V \times f^{-1}(y)$ such that the following diagram commutes:*

$$\begin{array}{ccc} f^{-1}(V) & \xleftarrow[\Psi]{} & V \times f^{-1}(y) \\ \downarrow f & \swarrow \pi & \\ V & & \end{array}$$

Corollary 12. *Let $f : M \to N$ be a proper submersion, N connected, then any two fibers $f^{-1}(y)$ are diffeomorphic (but not canonically so–what's the problem? the isomorphism in Ehresmann's theorem is by no means unique).*

Such maps are "differentiable fiber bundles". Spirit of the story: differentialbe manifolds have no moduli. Ehresmann theorem uses the first two techniques we mentioned above.

Let's consider the spacial case of $f : U \to V$ a proper submersion, $y \in V$ regular value, $U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^n$ open. At each point $x \in f^{-1}(y)$, $\mathbb{R}^m = \ker(Df_x) \oplus K_x$ (the orthogonal complement). $Df_x : K_x \xrightarrow{\cong} \mathbb{R}^n$. Hence get a basis $(e_1(x), \dots, e_n(x))$ of K_x by pullback. Consider $f^{-1}(y) \times \mathbb{R}^n \to \mathbb{R}^m$ given by $(x, t) \mapsto (x + t_1e_1(x) + \ldots + t_ne_n(x))$. The derivative of this is an isomorphism at each point of $f^{-1}(y) \times \{0\}$. In fact, it gives a diffeomorphism ψ between a neighborhood of $f^{-1}(y) \times \{0\} \subset f^{-1}(y) \times \mathbb{R}^n$ and a neighborhood of $f^{-1}(y) \subset U$. Now $f \circ \psi(x, 0) = y$. $D(f \circ \psi)_{(x,0)}|_{\{0\}\times\mathbb{R}^n}$ is an isomorphism. It follows that $f \circ \psi|_{\{x\}\times\mathbb{R}^n}$ is a local diffeomorphism near $t = 0$. Now invert all those diffeomorphisms (and observe that these vary continuously as x varies) to get Ψ.

Now consider the case of $f : M \to V \subset \mathbb{R}^n$ open, and $M \subset U \subset \mathbb{R}^m$ a submanifold for some U open, f is a proper submersion. At $x \in f^{-1}(y)$, $TM_x = T(f^{-1}(y))_x \oplus K_x$, the second being the orthogonal in TM_x. Then $Df_x : K_x \xrightarrow{\cong} \mathbb{R}^n$, get a basis $(e_i(x))$ of K_x. Recall that we can choose U so that there is a smooth retraction $r : U \to M$. Define $\psi(x, t), x \in f^{-1}(y), t \in \mathbb{R}^n$ small, to be $r(x + \sum_i t_ie_i(x))$. The rest is the same as before. Now note that we have:

Theorem 13.2 (Whitney)**.** *Every n-manifold can be embedded as a submanifold of $\mathbb{R}^{2n+1}$.*

The general Ehresmann then follows (but this uses all three techniques above.)

Now recall the partition of unity (we did this as a subset of $\mathbb{R}^n$, and the general proof is the same):

Theorem 13.3. *Let M be a manifold, (U_α), $\alpha \in A$ an open cover, then there are functions $(\psi_\beta), \beta \in B$, $\psi_b : M \to [0, \infty)$ such that $\operatorname{supp}(\psi_\beta)$ is contained in some U_α, and locally only finitely many ψ_β are nonzero, and $\sum_\beta \psi_\beta = 1$.*

Corollary 13. *For any M, there is a proper bounded below function $f : M \to \mathbb{R}$.*

Proof. Take $(\psi_i), i \in \mathbb{N}$ a countable partition of unity such that $\operatorname{supp}(\psi_i)$ is compact. Define $f(x) = \sum_{k=1}^{\infty} k \cdot \psi_k(x)$. If $x \notin \operatorname{supp}(\psi_1) \cup \ldots \cup \operatorname{supp}(\psi_n)$, then $f(x) \geq \sum_{k=n+1}^{\infty} k \cdot \psi_k(x) \geq (n + 1) \sum_{k=n+1}^{\infty} \psi_k(x) = n + 1$, hence properness. □

Remark 17. *Useful in combination with Sard to "decompose" a non-compact manifold into parts that are separated by compact pieces. For instance, consider the universal cover of $T^n\#M$, $\pi_1(M) = *$. It looks like an infinite $\mathbb{R}^n$ with a M at each lattice point, and this sort of cut it into concentric "balls" in that $\mathbb{R}^n$.*

Let's now consider Sard's theorem and its generalizations.

Proposition 5. *$f : M^m \to N^n$, $n > 2m$, then f can be perturbed to an injective immersion $\tilde{f}$, i.e. given $\Delta_N \subset V \subset N \times N$, V open in $N \times N$, we can ask that $(f(x), \tilde{f}(x)) \in V$ for all x. Or, more generally, given $M \times \Delta_N \subset U \subset M \times N \times N$ open, can ask that $(x, f(x), \tilde{f}(x)) \in U$ for all x.*

Principle of Proof. (Say for injectivity) Start with $F : P \times M \to N$ where $p_0 \in P$, $F(p_0, x) = f(x)$. Every value of P gives some way of deformation. This should be large enough that $P \times (M \times M) \setminus \Delta_M \xrightarrow{\Phi} N \times N$ given by $(p, x_1, x_2) \mapsto (F(p, x_1), F(p, x_2))$ is transversal to Δ_N. Then $\Phi^{-1}(\Delta_N) \subset P \times ((M \times M) \setminus \Delta_M)$ is a submanifold of codimension n. Take $\Phi^{-1}(\Delta_N) \to P$ and let p be a regular value close to p_0, then $\Phi^{-1}(\Delta_N) \cap \{p\} = \emptyset$ is empty by dimension argument. Set $\tilde{f} = F(p, \bullet)$. □

How to get large enough P? If M is compact, we can do partition of unity and locally allow finitely many degrees of perturbation. If M is non-compact, we have to be careful of maintaining finite-dimensionality: this is okay because for far enough open sets, we can recycle the degrees of perturbation. Or, an easier way is to take the space of all maps, but then we need to use Sard-Smale, and we have to be slightly careful about using Banach spaces (because C^∞ is not Banach; but it has dense subspaces that are).

Problem 14. *$f : M \to N$ smooth, M compact, show that any sufficiently small perturbation of f is homotopic to f.*

14 March 30

Theorem 14.1 (Homotopy Lifting)**.** *Let $f : M \to N$ a proper submersion. Given a family $(g_t)_{t\in\mathbb{R}}$ of diffeomorphisms of N, $g_0 = id$, there is a family $(h_t)_{t\in\mathbb{R}}$ of diffeomorphisms of M, $h_0 = id$, such that*

$$\begin{array}{ccc} M & \xrightarrow{h_t} & M \\ \downarrow f & & \downarrow f \\ N & \xrightarrow{g_t} & N \end{array}$$

In general, diffeomorphisms aren't linear, so it's hard to do linear combinations (e.g. partition of unity) with them. One way to solve this is to differentiate them, get a vector field, linearly combine the vector fields, and integrate them. Let's therefore talk about vector fields.

Definition 26 (Vector Field)**.** *(Physicist's Definition) Let M be a manifold. A vector field X on M is a collection of vector fields in local charts which are related by appropriate transformations.*

More concretely, let $\varphi_\alpha : \mathbb{R}^n \supset V_\alpha \to U_\alpha \subset M$. Let $\{X_\alpha : V_\alpha \to \mathbb{R}^n\}$ be the local vector fields. Suppose $V_\alpha \xrightarrow{\varphi_\alpha} M, V_\beta \xrightarrow{\varphi_\beta} M$, then $D(\varphi_\beta^{-1} \circ \varphi_\alpha)_x(X_{\alpha,x}) = X_{\beta,(\varphi_\beta^{-1}\circ\varphi_\alpha)(x)}$. This style of definition has no problem except one has to add a condition for two vector fields to be the same (namely, if you can put them together and still get a vector field).

Given a time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ on M, assuming that our vector fields are compactly supported (or in some other way prevented from reaching infinity), we get a family of diffeomorphisms $(\varphi_t)_{t\in\mathbb{R}}$, $\varphi_0 = id$, such that $\dfrac{\partial \varphi_t}{\partial t}(x) = X_t(\varphi_t(x))$. (This is not quite well-defined; use local chart definition.) In fact, once we solve it on each local chart, the overlap automatically agrees. Conversely, (φ_t) determines X_t.

Now let's prove isotopy lifting.

Proof Sketch. Start with family of diffeomorphisms (g_t) on N, we get a time dependent vector field X_t on N, which yields a time-dependent vector field on M (it is this step we explain below), which finally gives diffeomorphisms (h_t).

Now, look at a local chart $\mathbb{R}^m \supset U \xrightarrow{f} V \subset \mathbb{R}^n$, given $X_t : V \to \mathbb{R}^n$, want $Y_t : U \to \mathbb{R}^m$ such that $(*) Df_x(Y_{t,x}) = X_{t,f(x)}$. This way (h_t) is prevented from going to infinity because we have control on (g_t), and because the map is proper. We can locally find Y such that $(*)$ is satisfied by the local submersion theorem (Hmm?). Moreover, $(*)$ survives convex combinations, which means we can use partitions of unity. □

This tells us another way of showing that two fibers in a connected component are the same: construct a vector field moving from one to another, and lift.

Note that this gives another proof of Ehresmann's theorem. let $f : M \to N$, fix $y \in Y$ and families of diffeomorphisms on N: $\varphi_t^{(1)}, \varphi_t^{(n)}$ (so that they move in all possible directions, say each going one direction), then lift them to M, and use it to construct a local diffeomorphism between $f^{-1}(y) \times \mathbb{R}^n$ and M.

Now let's talk about manifolds with boundary.

Definition 27. *A topological manifold with boundary is a (Hausdorff, second countable) space locally homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times [0, \infty)$.*

Definition 28. *Let $A \subset \mathbb{R}^n$ be any subset. A function $f : A \to \mathbb{R}$ is smooth if there is an open $U \subset \mathbb{R}^n, A \subset U$ and a smooth function $g : U \to \mathbb{R}$ such that $g|_A = f$.*

This definition may be a bit eyebrow-raising because it doesn't seem to be local at the first sight. In fact it is. Here's an alternative definition:

Definition 29. *$f : A \to \mathbb{R}$ as before. f is smooth if for each $x \in A$ there is an open $U \subset \mathbb{R}^n$ containing x and a smooth $g : U \to \mathbb{R}$, such that $g|_{U\cap A} = f|_{U\cap A}$.*

Problem 15. *Show that the two definitions agree.*

Definition 30. *Let M be a topological n-manifold with boundary. A differentiable atlas is a collection of maps $(\varphi_\alpha : V_\alpha \xrightarrow{\cong} U_\alpha)$, where $(U_\alpha$ is an open cover of M, the V_α are open subsets of of $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times [0, \infty)$, and the φ_α are homeomorphism such that $V_\alpha \cap \varphi_\alpha^{-1}(U_\beta) \xrightarrow{\varphi_\beta^{-1}\varphi_\alpha} V_\beta \cap \varphi_\beta^{-1}(U_\alpha)$ is smooth for all (α, β). Note that smoothness at the boundary point requies a local smooth extension to an open subset o $\mathbb{R}^n$.*

Example 29. *Let M be a manifold, $f : M \to \mathbb{R}$ smooth, and a a regular value. Then $\{x \in M : f(x) \le a\}$ is a manifold with boundary.*

Note that by restricting the charts, the boundary of a manifold with boundary is a manifold of dimension one lower.

Theorem 14.2 (Collar Neighborhood Theorem)**.** *Let M be a differentiable manifold with boundary. Then there is an open subset $U \subset M$ containing ∂M, an open subset $V \subset \partial M \times [0, \infty)$ containing $\partial M \times \{0\}$, and a diffeomorphism $c : V \to U$ such that $c|_{\partial M\times\{0\}} = \mathrm{id}_{\partial M}$.*

Equivalent version:

Theorem 14.3. *Let M be a differentiable manifold with boundary. Then there is an open subset $U \subset M$ containing ∂M, and a diffeomorphism $c : \partial M \times [0, \infty) \to U$ such that $c|_{\partial M\times\{0\}} = \mathrm{id}_{\partial M}$.*

This is obvious locally, and of course it is not unique. The idea is as usual: partition of unity. Of course same problem: diffeomorphism isn't linear. Solution: go to vector fields.

Sketch of Proof. Construct a vector field pointing inwards along ∂M, by using a partition of unity to combine them, then integrate. □

Application 17. *Let M and N be two manifolds with boundary, together with a diffeomorphism $f : \partial M \to \partial N$. Define $O = M \sqcup N/\{x \sim f(x), x \in \partial M\}$, then this is a topological manifold without boundary.*

Note that O doesn't have a unique differentiable structure. (Differentiable on two sides and agree on boundary isn't enough to say everywhere differentiable.) To get a differentiable structure, choose collar neighborhoods $\partial M \times (-\infty, 0] \supset V \xrightarrow{c} M$ and $\partial N \times [0, \infty) \supset W \xrightarrow{d} N$, and identify $V \sqcup W/\{(0, x) \sim (0, f(x))\} \xrightarrow{c\sqcup d} O$. But note that the set sites in $\partial N \times \mathbb{R}$ by $(x, t) \mapsto (f(x), t)$ for $x \in V$, and id on $x \in W$, then this gives a map from an open neighborhood of $\partial N \times 0 \subset \partial N \times \mathbb{R}$ to O, which we can use to define the differentiable structure near the "scam." This differentiable structure is unique up to diffeomorphism.

Example 30. *Connected sum $M\#N$ can be viewed as gluing together $(M \setminus B)$ and $(N \setminus B)$ for a small open ball B.*

Definition 31. *Two compact manifolds M_1, M_2 are called (co)bordant if $M_1 \sqcup M_2$ is the boundary of a compact $(n+1)$-manifold.*

Remark 18. *Any M is cobordant to itself: $M \sqcup M = \partial(M \times [0,1])$.*

Remark 19. *Bordism is transitive, by gluing.*

Thus bordism is an equivalence relation.

Definition 32. *The bordism ring $\Omega_* = (\Omega_n)_{n \geq 0}$ is the set of equivalence classes of compact manifold under bordism, with disjoint union as addition ($[M] + [M] = [\emptyset]$) and product as multiplication ($[M] \times [*] = [M]$).*

Example 31. $\Omega_0 \cong \mathbb{F}_2$, $\Omega_1 = 0$.

We'll see that higher bordisms will in some way serve as generalizations for numbers to higher dimensions.

15 April 4

Recall that $\Omega_0 = \mathbb{Z}/2$, $\Omega_1 = 0$. Today we prove that $\Omega_2 = \mathbb{Z}/2$.

Remark 20. *Recall that $[M] + [M] = 0 \in \Omega_*$, because $M \sqcup M = \partial(M \times I)$. It is also true that $[M \# N]$ (connected sum) is equal to $[M] + [N]$ due to the standard bordism between these manifolds.*

The classification theory of surfaces tells any closed surface is diffeomorphic to a connected sum of T^2s and $\mathbb{RP}^2$s. We know $T^2 = \partial(S^1 \times D^2)$, hence Ω_2 is either 0 (if $\mathbb{RP}^2$ is an embedding) or $\mathbb{Z}/2$ (otherwise).

Let M be a manifold with boundary. An orientation of M determines an orientation of ∂M. Example: $\partial(\bullet \to \bullet) = \bullet_{-1}\ \bullet_{+1}$. The general definition is by restricting charts (here, the convention is that charts along ∂M are modelled on $[-\infty, 0] \times \mathbb{R}^{n-1}$). Why does this work? Suppose we have transition maps $(-\infty, 0) \times \mathbb{R}^{n-1} \xrightarrow{\varphi} (-\infty, 0) \times \mathbb{R}^{n-1}$ if φ is a positively oriented diffeomorphism ($\det D\varphi > 0$), then the same holds for $\varphi|_{\{0\} \times \mathbb{R}^{n-1}}$, because at $x \in \{0\} \times \mathbb{R}^{n-1}$, $(D\varphi)_x = \begin{pmatrix} > 0 & 0 \\ * & * \end{pmatrix}$.

Two compact oriented manifolds M_1, M_2 are oriented co(bordant) if $M_1 \sqcup (-M_2)$ is the boundary of a compact oriented manifold. We can likewise define the oriented bordism ring Ω_*^{SO}. For instance, $\Omega_0^{SO} = \mathbb{Z}$, $\Omega_1^{SO} = 0$, $\Omega_2^{SO} = 0$ (by the classification of surfaces), $\Omega_3^{SO} = 0$ (needs a lot of work), $\Omega_4^{SO} \neq 0$) (in fact $\mathbb{Z}$ but we'll show today that $\Omega_4^{SO} \to \Omega_4^{O}$ is nontrivial).

Example 32. *Let $\Gamma \subset U(n)$ be a finite subgroup. Assume that Γ acts freely on $\mathbb{C}^n \setminus \{0\}$. Then Γ acts freely on $S^{2n-1} \subset \mathbb{C}^n$, and $M = S^{2n-1}/\Gamma$ is an oriented manifold. This is called the link of the singularity $0 \in \mathbb{C}^n/\Gamma$. Then $[M] = 0 \in \Omega_{2n-1}^{SO}$ by resolution of singularities, as $M = \partial(\widetilde{D^{2n}/\Gamma})$. In particular, all odd-dimensional real projective spaces are bound (Take $\Gamma = \{\pm 1\}$; they are orientable and bound as orientable manifolds).*

And let's review some basic topological invariants.

Degree $f : M^m \to N^m$ map between compact manifolds of the same dimension, N connected (or a proper map between not necessarily compact manifolds), $\deg_{\mathbb{Z}/2}(f) = \# f^{-1}(y) \in \mathbb{Z}/2$ for $y \in N$ a regular value.

If M, N are oriented, can define the $\mathbb{Z}$-degree $\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} \det(Df_x) \in \mathbb{Z}$ where the determinant is in oriented local charts. This is well defined and is homotopy invariant.

Consider $f : M^m \to N^n$ (again, compact manifolds or proper map, N connected). How does the preimage of a proper value vary? If we have a submersion, Ehresmann's theorem says they are all the same; in general this of course is false, but:

Theorem 15.1. *Let y be a regular value. Then $[f^{-1}(y)] \in \Omega_{m-n}$ is independent of y and a homotopy invariant of f.*

In this way, bordism classes generalize the notion of "numbers" as one count solutions.

Outline of Proof. Given two regular values y_0, y_1, choose a path $c(0) = y_0, c(1) = y_1$. (Be careful: we can't assume all points along this line are regular value; for instance, singular points can form a hypersurface that we have to cross.) We can achieve (by perturbing c) that this is transverse to f, in the sense that $[0,1] \times M \xrightarrow{\varphi} N \times N$ given by $(t,x) \mapsto (c(t), f(x))$ is transverse to the diagonal Δ_N. Then $\varphi^{-1}(\Delta_N) = \{(t,x) : c(t) = f(x)\}$ is a manifold with boundary where $t = 0, 1$. The boundary is $f^{-1}(y_0) \sqcup f^{-1}(y_1)$. □

Theorem 15.2. *If in addition M and N are oriented, then $[f^{-1}(y)] \in \Omega^{SO}_{m-n}$ has the same properties. Note that the preimage acquires an orientation by working in local oriented charts.*

Now let M^m a compact manifold, $f : M^m \to M^m$ (or can ask for any M, then $f(M)$ needs to be compact). We $L(f) = \sum_{f(x)=x} \operatorname{sgndet}(1 - D_x f)$. (Assuming the det is not zero oin local charts). Check this is well-defined: suppose transition is $\tilde{f} = \psi^{-1} f \psi$, then $\det(1 - D\tilde{f}) = \det(1 - D\psi^{-1} Df D\psi) = \det(D\psi^{-1} D\psi - D\psi^{-1} Df D\psi) = \det(1 - Df)$. This is independent of orientability. To define it in general, have to perturb f such that the determinant is nonzero at all fixed points. $L(f)$ is independent of the perturbation used to define it, and is homotopy invariant.

Example 33. *If f has no fixed points, $L(f) = 0$.*

Lemma 21. *M, N compact manifolds, $f, g : M \to N$, then $L(fg) = L(gf)$.*

Definition 33. *M compact manifold. The euler characteristic $\chi(M) = L(\mathrm{id}_M)$.*

Proposition 6. *$\chi(M)$ is a homotopy invariant.*

Proposition 7. *If $f : M \to N$ is a d to 1 covering, then $\chi(M) = d\chi(N)$.*

For instance, S^1 must have $\chi = 0$, because S^1 is a d to 1 of itself for any d.

Proposition 8. *If $f : M \to N$ is a submersion and N is connected, $\chi(M) = \chi(N)\chi(f^{-1}(y))$ for any y.*

Example 34. *$\chi(\mathbb{CP}^{n-1}) = n$. Remember $\mathbb{CP}^{n-1}$ are lines in $\mathbb{C}^n$, so any $A \in \mathrm{GL}_n(\mathbb{C})$ acts on it, and the fixed points correspond precisely to the eigenvectors. Deform $A = 1$ to $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ for pairwise distinct $\lambda_1, \dots, \lambda_n$. A computation in local coordinates shows that all n fixed points contribute $+1$.*

Note that this proves that complex matrices have eigenvectors.

Example 35. *$\chi(\mathbb{RP}^{n-1}) = 0$ if n is even, and 1 if n is odd (choose matrix in $\mathrm{GL}^+_n(\mathbb{R})$ with just one eigenvalue, compute in local coordinates).*

Thus $\chi(S^{n-1}) = 0$ if n even, 2 if n odd.

Theorem 15.3. *M compact manifold, $\dim(M) = n$ odd, then $\chi(M) = 0$.*

Proof. Perturb id_M to $f : M \to M$, which will be a diffeomorphism (being a diffeomorphism is an open condition), so $L(f) = \sum_{x=f(x)} \operatorname{sgn}\det(1 - Df_x)$, and

$L(f^{-1}) = \sum_{x=f^{-1}(x)} \operatorname{sgn}\det(1 - Dxf^{-1}) = \sum_{f(x)=x} \operatorname{sgn}\det(D_x f^{-1}) \det(D_x f - 1)$. Now $\det(D_x f^{-1}) \approx 1$, so we get $\sum_{f(x)=x} \operatorname{sgn}\det(D_x f - 1) = (-1)^n L(f)$. But $\chi(M) = L(f) = L(f^{-1})$, so we get 0 when n is odd. □

Theorem 15.4. *M compact, $\dim(M)$ even, $\chi(M)$ odd, then M is not the boundary of a compact $(n+1)$-manifold.*

We'll prove this later. (This was proved in April 11's class.)

16 April 6

Recall we defined basic topological invariants: degree and Lefschetz number (Euler characteristic).

Remark 21. *Let M be a compact manifold with boundary. Using partitions of unity and collar neighborhood, one can define $f_t : M \to M$ for $t \geq 0$, $f_0 = \mathrm{id}$, $f_t(M) \subset M \setminus \partial M (t > 0)$. This corresponds to a vector field that "pushes boundary inwards."*

Now given $f : M \to M$, one can define $L(f) = L(f|_{M\setminus\partial M} \circ f_t)$ for $t > 0$, where $f_t \circ f$ maps $M \setminus \partial M$ to a relatively compact subset of $M \setminus \partial M$. In particular, this defines $\chi(M) = L(f_t)(t > 0)$.

Intersection Number $f_1 : M_1 \to N, f_2 : M_2 \to N$, M_1, M_2 are compact, and $\dim(N) = \dim(M_1) + \dim(M_2)$. Perturb f_1, f_2 such that (f_1, f_2) is transvsers to the diagonal Δ_N, and the intersection number $f_1 \circ_{\mathbb{Z}/2} f_2 \in \mathbb{Z}/2$ is defined as $|(f_1, f_2)^{-1}\Delta_N| \in \mathbb{Z}/2$, i.e. how many times the images intersect.

If all manifolds are orientable, can define the $\mathbb{Z}$-intersection number similarly.

Remark 22. *If M is compact and oriented, and $f : M \to M$, define $\gamma_f : M \to M \times M$ by $\gamma_f(x) = (x, f(x))$ and $\delta(x) = (x, x)$. Then $\gamma_f \cdot \delta = L(f)$. In particular, $\delta \cdot \delta = \chi(M)$.*

Note that Euler characteristic / Lefschetz number didn't ask for orientability; in fact, one can get away with something slightly weaker than all manifolds to be (separately) orientable, and in that setting we can get them without orientability condition.

Problem 16. *Show that there is no map $f : S^2 \to T^2$ with $\deg_{\mathbb{Z}/2}(f) = 1$. (Probably use intersection number). (Note there is a map of degree one in the other direction: take a small disk of T^2 and map it to $S^2 \setminus \{*\}$, and map the rest to $\{*\}$).*

Generalizations $f : M \to N$, N connected, f proper, $[f^{-1}(y)] \in \Omega_{\dim(M)-\dim(N)}$ for a regular value of y. This is an invariant that generalizeds $\deg_{\mathbb{Z}/2}(f)$.

Remark 23. *In particular, if for some regular y, $[f^{-1}(y)] \in \Omega_*$ is nontrivial, then f must be onto.*

If M, N are oriented, can get $[f^{-1}(y)] \in \Omega^{SO}_{\dim(M)-\dim(N)}$, which generalizes the integer degree.

Now if $f_1 : M_1 \to N, f_2 : M_2 \to N$ for compact M_1, M_2, then after perturbing, $[(f_1, f_2)^{-1}(\Delta_N)] \in \Omega_{\dim(M_1)+\dim(M_2)-\dim(N)}$ is an invariant. If M_1, M_2, N are oriented, can get the integer version as well.

What about Lefschetz number / Euler characteristic?

Let $f : M \to N$ be a smooth proper submersion. Suppose $g : M \to M$ is a fiberwise self-map i.e. $gf = f$, under a suitable undegeneracy condition, $\mathrm{Fix}(g)$ is a submanifold of $\dim(N)$. The Dold-Lefschetz fixed point index is $[\mathrm{Fix}(g)] \in \Omega_{\dim(N)}$. If N is oriented, so is $\mathrm{Fix}(g)$ and we get $[\mathrm{Fix}(g)] \in \Omega^{SO}_{\dim(N)}$. In particular, this can be used to define an Eulcer characterstic of $f : M \to N$ which is a bordism class of dimension $\dim(N)$. (In reality, only N necessarily needs to be a manifold; say M can be a fiber bundle or something like that.)

What is this notion of nondegeneracy? Local picture is $\mathbb{R}^m \xrightarrow{g} \mathbb{R}^m, f : \mathbb{R}^m \to \mathbb{R}^n$ and $gf = f$ (why are we staying on the same chart? Note we're only talking about near fixed points). Suppose by submersion theorem that $f(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$. Then we require that we have $g(x_1, \ldots, x_m) = (x_1, \ldots, x_n, h(x_1, \ldots, x_m))$ such that 0 is a regular value of $(x_1, \ldots, x_m) \mapsto (x_{n+1}, \ldots, x_m) - h(x_1, \ldots, x_m)$. Note that preimages of zero are the fixed points.

This is a weaker condition that to require the fixed point is fiberwise nondegenerate (which is impossible).

Remark 24. *If M, N are oriented, one can think of this as an intersection number in $M \times_f M$. This generalizes what we said above.*

Bordism with Target Space Fix a manifold X. Consider $(M, f), M$ compact, $f : M \to X$. Two such pairs $(M_1, f_1), (M_2, f_2)$ are cobordant if there is a compact manifold with boundary N and $j : N \to X$ such that $\partial N = M_1 \sqcup M_2$ ($-M_2$ in the oriented case), $g|_{M_1} = f_1, g|_{M_2} = f_2$. ($X$ can also be an open set or other relaxed conditions; for now let's use manifold.) This is an equivalence relation, and gives abelian groups $\Omega_*(X)$ or $\Omega^{SO}_*(X)$. We shall focus on the unoriented case, where twice of anything is zero.

There is no longer a ring structure but we still have the *exterior product* $\Omega_*(X) \otimes \Omega_*(Y) \to \Omega_*(X \times Y)$.

Functoriality Given $\varphi : X \to Y$, get $\varphi_* : \Omega_*(X) \to \Omega_*(Y)$ and this is a homotopy invariant of φ.

Mayer-Victoris Sequence Given a smooth function $\psi : X \to \mathbb{R}$, define $X_- = \psi^{-1}((-\infty, 1))$ and $X_+ = \psi^{-1}((-1, \infty))$, $X_+ \cap X_- = \psi^{-1}((-1, 1))$. Then there is a long exact sequence

$$\ldots \to \Omega_*(X_+ + \cap X_-) \xrightarrow{\text{inclusion},-\text{inclusion}} \Omega_*(X_+ \oplus \Omega_*(X_-)) \xrightarrow{\text{inclusion},\text{inclusion}} \Omega_*(X) \to \Omega_{*-1}(X_+ \cap X_-) \to \ldots$$

Remark 25. *This is an (extraordinary) homology theory. It is a theorem that* $\Omega_*(X) \cong H_*(X; \Omega_*) \cong H_*(X) \otimes \Omega_*$. *(But this is not true in the oriented case).*

Inverse (shriek) Functoriality For $\varphi : X \to Y$ a proper map, we get $\varphi^! : \Omega_*(Y) \to \Omega_{*+\dim(X)-\dim(Y)}(X)$.

Example 36. $\dim(X) = \dim(Y)$, *Y connected,* $\varphi^!([* \to Y]) \in \Omega_0(X)$ *recovers the degree of* φ.

To define this, given $f : M \to Y$, perturb $(\varphi, f) : X \times M \to Y \times Y$ so that it is transverse to Δ_Y. Then the preimage of the diagonal is a compact submanifold of $X \times M$, we map it to X by projection.

Intersection Product We have a map $\Omega_i(X) \otimes \Omega_j(X) \to \Omega_{i+j-\dim(X)}(X)$, which can be defined using diagonal embedding $\Omega_*(X) \otimes \Omega_*(X) \to \Omega_*(X \times X) \xrightarrow{\delta^!} \Omega_{*-\dim(X)}(X)$ for δ the diagonal map.

Remark 26. *Shriek functoriality and intersection product use the fact that the target space is a manifold (Intersection product: Poincare, cup product, poincare again) (Shriek: poincare, pullback, poincare again).*

Remark 27. *Every mod 2 homology can be represented as bordism; this is false for integral homologies.*

17 April 11

Let's prove the theorem we forgot to prove.

Theorem 17.1. *If M is a compact even-dimensional manifold, $\chi(M)$ is odd, then M is not the boundary of a compact manifold.*

Proof. Recall that $\chi(M) = L(\mathrm{id}_M)$. To compute it, we choose some $f : M \to M$ which is homotopic to the identity and has nondegenerated fixed points. Then $\chi(M) = L(f) = |\mathrm{Fix}(f)| \bmod 2$. Assume $M = \partial N$; define $P = N \cup_M N$. Choose a map $f : M \to M$ as before, and extend it to a map $g : N \to N$, such that:

- g is homotopic to the identity through maps that take $\partial N = M$ to itself.
- g has nondegenerate fixed points.
- for some choise of collar neighborhood $M \subset U \subset N \xrightarrow{\varphi} \{0\} \times M \subset [0, \infty) \times M$, we have $g(\varphi^{-1}(r, x)) = f(x)$ if (r, x) is sufficiently closed to the boundary.

Now define $h : P \to P$ to be two compies of g. This is homotopic to the identity, has nondegenerate fixed points, and $|\mathrm{Fix}(h)| = 1 \bmod 2$ (things on sides are symmetric). Contradiction (recall that P is odd dimensional), thus $|\mathrm{Fix}(f)| = \chi(M) \bmod 2$ must be even. □

Vector bundles

Definition 34. *Let $\pi : E \to M$ be a smooth map. Suppose that each fiber $F_x = \pi^{-1}(x)$ comes with the structure of a (finite-dimensional) real vector space. Suppose also that those structures are locally trivial, i.e. for any $x \in M$ there is an open neighborhood $U \subset M$ and a diffeomorphism $E_x \times U \to \pi^{-1}(U)$ that is compactible with the fiberwise vector space structure. Then we call E a real vector bundle over M.*

Remark 28. *This implies that π is a submersion.*

A section of $\pi : E \to M$ is a smooth map $s : M \to E$, $\pi(s(x)) = x$ $\forall x$.

Lemma 22. *The space of all sections $\Gamma(E)$ is canonically a module over $C^\infty(M)$, the ring of real-valued functions.*

A homomorphism of vector bundles $E \to F$ is a smooth map ϕ that takes fibre to fibre and is linear on each fibre.

Lemma 23. *A vector bundle map $\phi : E \to F$ induces a $C^\infty(M)$-module map $\Gamma(E) \to \Gamma(F)$.*

Proposition 9. *The category of vector bundles of bounded rank over M is equivalent to that of finitely generated projective $C^\infty(M)$ modules.*

Example 37. *The tangent bundle $TM \to M$. The fibre is the tangent space over x. We use local charts to check the triviality condition.*

Transition Maps Take $\pi : F \to M$ a vector bundle of rank r, a cover $M = \bigcup_{\alpha \in A} U_\alpha$, and local trivializations $\varphi_\alpha : \mathbb{R}^r \times U_\alpha \to \pi^{-1}(U_\alpha)$. Then $\varphi(\beta)^{-1}\varphi_\alpha(v, x) = (A_{\beta\alpha}(x)v, x)$ for some linear map $A_{\beta\alpha}(x)$, and that is in turn given by some smooth map $A_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$. These maps satisfy the cocycle condition $A_{\gamma\beta}A_{\beta\alpha} = A_{\gamma\alpha}$. Conversely, from an open cover and a collection of $(A_{\beta\alpha})$ we can construct a vector bundle.

Remark 29. *A collection $(A_{\beta\alpha})$ is a Cech 1-cocycle with values in the topological group $\mathrm{GL}(r, \mathbb{R})$. In fact, rank one vector bundles up to isomorphism are isomorphic to $\check{H}^1(M, \mathrm{GL}(r, \mathbb{R}))$.*

Example 38. *Consider $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$, which induces $\ldots \to H^1(M, C) \to H^1(M, \mathbb{C}^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C}) \to \ldots$; the first and last terms vanish, and the second is $\mathrm{GL}(1, \mathbb{C})$, which tells us the usual knowledge that the isomorphism class of complex line bundle is specified by the first Chern class.*

Example 39. *Transition functions for TM are given by the derivatives of coordinate changes.*

Constructions with vector bundles

- Direct sum $E \oplus F$, where $(E \oplus F)_x = E_x \oplus F_x$. Strictly speaking, the manifold underlying $E \oplus F$ is $\{(v, w) \in E \times F : (v, w) \text{ lie over the same point of } M\}$.
- Dual space E^*, where $E_x^* = \mathrm{Hom}_\mathbb{R}(E_x, \mathbb{R})$.
- Tensor product $E \otimes F$.
- Internal hom $\underline{\mathrm{Hom}}(E, F) = E^* \otimes F$.
- Symmetric product $\mathrm{Sym}^r(E)$ and exterior product $\Lambda^r(E)$.

Remark 30. *More systematically, any Lie group homomorphism $GL_(r_1, \mathbb{R}) \times \ldots \times GL(r_k, \mathbb{R}) \to \mathrm{GL}(r, \mathbb{R})$ gives a way of constructing vector bundle of rank r from ones of rank $(r_1, \ldots, r_k)$.*

In particular, if E has rank r, we have the determinant line $\det(E) = \Lambda^r(E)$ which is a line bundle (rank 1 vector bundle).

Lemma 24. *Let $\phi : E \to F$ be a map of vector bundles such that $\mathrm{rank}(\phi_x)$ is locally constant. Then there are local trivializations of E, F around any point such that ϕ is locally constant.*

Corollary 14. *If $\phi : E \to F$ has a locally constant rank, there are natural vector bundles $\ker(\phi) \subset E, \mathrm{im}(\phi) \subset F$, and $\mathrm{coker}(\phi) = E/\mathrm{im}(\phi)$.*

Corollary 15. *If $E \subset F$ is a vector subbundle, there is a natural quotient bundle F/E with map $F \twoheadrightarrow F/E$.*

Consider a short exact sequence of vector bundles

$$0 \to E \xrightarrow{\phi} F \xrightarrow{\psi} G \to 0$$

Then $E \cong \ker(\psi), G \cong \mathrm{coker}(\phi)$ canonically. Unfortunately we do not have an abelian category due to the restriction of locally constant rank (one can't get rid of this by restricting morphisms–note that being locally constant rank is not closed under composition).

Problem 17. *If $0 \to E \xrightarrow{\phi} F \xrightarrow{\psi} G \to 0$ is a short exact sequence of vector bundles, then there is a canonical isomorphism $\det(F) \cong \det(E) \otimes \det(G)$.*

18 April 13

A real line bundle is a real vector bundle $\pi : E \to M$ of rank 1. Fibres E_x carry structures of a 1-dimensional real vector space, in a way which is locally constant.

Lemma 25. *A real line bundle is trivial (isomorphic to the trivial bundle $\underline{\mathbb{R}}$) if and only if it has a nowhere vanishing section.*

Proof. For any E, sections correspond to vector bundle maps $\underline{\mathbb{R}} \to E$. A section which is nowhere zero thus corresponds to an injective vector bundle map. □

Recall that we have $E \otimes F$ the tensor product of vector bundles. $E^* \otimes F \cong \mathrm{Hom}(E, F)$ is the bundle of fiberwise linear maps. For a line bundle $L^* \otimes L \cong \mathrm{Hom}(L, L) \cong \underline{\mathbb{R}}$.

Corollary 16. *Isomorphism classes of line bundles with tensor form an abelian group* $\mathrm{Pic}_{\mathbb{R}}(M)$*. This acts by tensor on the set of isomorphism classes of vector bundles.*

Remark 31. *We can also consider complex vector bundles (or indeed left/right quaternion ones) over the real manifold M. Everything said so far applies to complex line bundles as well, which form an abelian group* $\mathrm{Pic}_{\mathbb{C}}(M)$*. Given a real vector bundle, one can form its complexification $E \to E \otimes_{\mathbb{R}} \mathbb{C} = E \oplus iE$ (two copies of E, with $\sqrt{-1}$ acting by $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$). In particular, we get a group homomorphism* $\mathrm{Pic}_{\mathbb{R}}(M) \to \mathrm{Pic}_{\mathbb{C}}(M)$.

Remark 32. *In fact,* $\mathrm{Pic}_{\mathbb{R}}(M) = H^1(M; \mathbb{Z}/2)$ *and* $\mathrm{Pic}_{\mathbb{C}}(M) = H^2(M; \mathbb{Z})$.

Unfortunately, tensor product of quaternion line bundles is not well-defined.

Problem 18. *Let $D \subset M$ be a hypersurface (a codimension 1 submanifold), construct a canonical line bundle L_D such that sections of L_D correspond to functions $M \to \mathbb{R}$ which vanish along D.*

To any line bundle $L \to M$ one can associated canonically a double covering $L^{\mathrm{or}} \to M$, where $L^{\mathrm{or}} = \{\zeta \in L : \zeta \neq 0\}$ (in its fiber) / $\mathbb{R}^+ \to M$. More generally, given a vector bundle E we can define $E^{\mathrm{or}} \to M$ by setting $E^{\mathrm{or}} = \{\text{bases } (\zeta_1, \dots, \zeta_r) \text{ of } E_x\}/\mathrm{GL}^+(r, \mathbb{R})$. The fibre of $E^{\mathrm{or}} \to M$ over x is the set of orientations of E_x.

Example 40. *For TM, $TM^{\mathrm{or}} = M^{\mathrm{or}}$ is the orientation cover of M.*

Definition 35. *An orientation of a vector bundle $E \to M$ is a continuous section of $E^{\mathrm{or}} \to M$ (for $E = TM$, this is the same as an orientation of M itself).*

Lemma 26. *There is a canonical isomorphism $E^{\mathrm{or}} \cong (\det(E))^{\mathrm{or}}$.*

Proof. Take $(\zeta_1, \dots, \zeta_r) \mapsto \zeta_1 \wedge \dots \wedge \zeta_r$. □

Corollary 17. *Given a short exact sequence of vector bundle $0 \to E \to F \to G \to 0$ an orientation of any two of them has an orientation of the third.*

Proof. Reduce to the canonical isomorphism $\det(F) \cong \det(E) \otimes \det(G)$. □

Pullback of vector bundle $f : M \to N$, given $E \xrightarrow{\pi} N$, defined $f^*E = \{(\zeta, x) \in E \times M \mid \pi(\zeta) = f(x)\}$, $f^*(E)_x = E_{f(x)}$.

Application 18. *$f : M \to N$, the derivative $Df_x : TM_x \to TN_{f(x)}$ is part of a vector bundle map $TM \xrightarrow{Df} f^*TN$. f is an immersion (submersion) if this is injective (onto).*

Corollary 18. *Let M, N be oriented manifolds. If $f : M \to N$ is a map, and y a regular value, then $f^{-1}(y)$ inherits a preferred orientation.*

Proof. Call $F = f^{-1}(y)$. Then $0 \to TF \hookrightarrow TM|_F \xrightarrow{Df} (f^*TN)|_F \to 0$, where $(f^*TN)|_F$ is the trivial bundle with fibre TN_y. □

Definition 36. *Let $N \subset M$ be a submanifold. The normal bundle of N in M is $\nu N = (TM|_N)/TN$, i.e. $0 \to TN \to TM|_N \to \nu N \to 0$.*

Application 19. *$f : M \to N$ a map, $O \subset N$ a submanifold. f is transverse to O iff at each point $x = f^{-1}(y), y \in O$, $TM_x \xrightarrow{Df_x} TN_y \to \nu O_y$ is surjective.*

Corollary 19. *M, N oriented, $f : M \to N$, $O \subset N$ oriented submanifold transverse to f, then $f^{-1}(O)$ inherits a preferred orientation.*

Proof. Call $P = f^{-1}(O)$, then $0 \to TP \to TM|_P \xrightarrow{\text{projection} \circ Df} f^*(\nu O) \to 0$. On the other hand, $0 \to TO \to TN|_O \to \nu O \to 0$. Now chase. □

Remark 33. *We don't need N and O to be oriented, it's enough if νO is oriented (i.e. O is co-oriented).*

That induces a co-orientation of P, since

$$\nu P \cong f^*(\nu O)$$

(This is important!) Then, if M is oriented, the co-orientation of P determines an orientation.

Application of partitions of unity

Proposition 10. *Given a surjective map of vector bundles $\phi : E \to F$, there is a map $\psi : F \to E$ such that $\phi\psi = \mathrm{id}_F$. This implies $\psi\phi$ is an idempotent endomorphism, so $E = E_0 \oplus E_1$ with $E_0 = \ker(\phi)$ and $E_1 \xrightarrow{\phi} F$.*

Corollary 20. *Given a short exact sequence of vector bundles $0 \to E \xrightarrow{\phi} F \xrightarrow{\pi} G \to 0$ there is an isomorphism $F \cong E \oplus G$ such that ϕ is inclusion, π is projection; in other words, the exact sequence splits.*

On complex geometry this is true for complex bundles on Stein manifolds, and algebraic vector bundles over affine varieties. Of course in neither case it is generally true.

Proof. ϕ is of locally constant rank, $M = \bigcup_\alpha U_\alpha$ and trivialization of $E|_{U_\alpha}, F|_{U_\alpha}$ so that ϕ is constant rank in each trivialization, Hence we get maps $\Psi_\alpha : F|_{U_\alpha} \to E|_{U_\alpha}$, Take a subordinate partition of unity, and the corresponding weighted sum of Ψ_α. □

Corollary 21. *If $N \subset M$ is a submanifold, $TM|_N \cong TN \oplus \nu N$ (non-canonically). This does not hold in algebraic geometry.*

Theorem 18.1 (Tubular Neighborhood Theorem)**.** *Let $N \subset M$ be a submanifold, then there is a diffeomorphism between a neighborhood of N in M and a neighborhood of the 0 section in νN.*

Proof Sketch. embed $M \subset \mathbb{R}^p$, choose a neighborhood $M \subset U \subset \mathbb{R}^p$ and retraction $r : U \to M$. $\nu N \hookrightarrow TM|_N \hookrightarrow \mathbb{R}^p \times N \xrightarrow{(\zeta,x) \mapsto \zeta + x} U \xrightarrow{r} M$, where the first is from $TM|_N$ splitting, and last map needs to restrict to some neighborhood. Easy computation shows this is a local diffeomorphism near each point of N. □

19 April 20

Application of partitions of unity to vector bundles

Definition 37. *A euclidean metric on a vector bundle $E \to M$ is a Euclidean metric on each vector space E_x, which varies smoothly in x (with respect to local trivializations).*

Remark 34. *One can find local trivializations in which the Euclidean metric is constant (equal to the standard metric on $\mathbb{R}^n$). (Use Gram-Schmidt, which makes smooth choices.)*

Application 20. *Let $E \to M$ be a rank 2 oriented bundle. Define $I_x : E_x \to E_x$ to be rotation by $\pi/2$ (i.e. the unique orthogonal automorphism satisfies $I_x^2 = -id$ and such that $(v, I_x v)$ is a positive oriented basis for all $v \neq 0$) This makes E into a complex rank 1 vector bundle. In fact, oriented Euclidean real rank 2 bundles correspond to complex hermitian rank 1 bundle.*

Application 21. *Let $0 \to E \xrightarrow{\iota} F \xrightarrow{\phi} G \to 0$ be a short exact sequence of vector bundles. If F has a Euclidean metric, there is a preferred splitting $F = \iota(E) \oplus \iota(E)^\perp$, where the second one is the fiberwise orthogonal complement.*

Lemma 27. *Every vector bundle admits a Euclidean metric.*

Proof. Partition of unity. (For complex vector bundles and hermitian metrics, the same applies.) □

Corollary 22. *The isomorphism classes of real oriented rank 2 bundles correspond bijectively to the isomorphism classes of complex line bundles.*

Corollary 23. *A real line bundle $L \to M$ is trivial iff the double cover $L^{\mathrm{or}} \to M$ is trivial.*

Proof. Suppose L^{or} is trivial and choose trivialization (orientation of L). Equip L with a Euclidean metric, then in each L_x there is a unique positively oriented vector of length 1. These vectors give you a nowhere vanishing smooth section. □

Remark 35. *In fact,* $\mathrm{Pic}_{\mathbb{R}}(M) = \{$*double covers of* $M\} =^{\text{if } M \text{ is connected}} \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2)$.

This says in particular that $\mathrm{Pic}_{\mathbb{R}}(M)$ is a $\mathbb{Z}/2$-vector space; in particular $L \otimes L$ is always trivial. But we know this: given L a Euclidean metric, $L \cong L^*$. Then $L \otimes L \cong L \otimes L^* \cong \mathrm{Hom}(L, L) = \underline{\mathbb{R}}$.

Corollary 24. *For any real vector bundle, $E \cong E^*$ non-canonically. For a complex vector bundle, $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C}) \cong \overline{E}$ (complex conjugate). Again this is not canonical.*

Sard's Theorem and Similar Transversality Techniques

Theorem 19.1. *Given a vector bundle $E \xrightarrow{\pi} M$, there is a section $M \xrightarrow{s} E$ which is transverse to the zero section. (In fact, this can be achieved by slightly perturbing any given section.)*

Proof Sketch. Construct an auxiliary manifold P and a section s of $q^*E \to P \times M$ (where $q : P \times M \to M$); This corresponds to a family of sections of M parametrized by P, which is transverse to the zero section. Then project $s^{-1}(0) \to P$ and choose a regular value. □

How to choose P? We can easy choose some large finite-dimensional parameter space. (e.g. Suppose M is compact, use partition of unity, and then choose all constant sections of all directions.) Or, choose P the space of all possible sections. (Again be careful with Banach.)

Note that in algebraic geometry we might not even have any section; but if the sheaf is generated by global sections, they an analogous statement still holds.

Corollary 25. *If $E \to M$ is a vector bundle with rank* $\mathrm{rank}(E) > \dim(M)$ *then $E \cong \underline{\mathbb{R}} \oplus \mathbb{F}$ for some F.*

Proof. We can find a section that is nowhere zero. (See Application 19; note that preimage of the zero section must be empty in this case, because the map cannot possibly be surjective.) That section yields an injective map $\underline{\mathbb{R}} \to E$. □

Example 41. *There are exactly two vector bundles of any rank r over the circle. (The trivial one and the unorientable one).*

The following is the counterpart of the Whitney embedding theorem for manifolds.

Theorem 19.2. *Let $E \to M$ be a vector bundle of rank r. Then there is an injective bundle map $E \to \underline{\mathbb{R}}^{r+\dim(M)}$.*

Proof. Step 1: suppose M is compact. Then we show that for any $E \to M$ there is an injective vector bundle map $E \to \mathbb{R}^N$, $N \gg 0$. Note that for any $x \in M$ there is a vector bundle map $\varphi_x : E \to \underline{\mathbb{R}}^n$ which is an isomorphism on the fibre at x. That map is an ismorphism also for fibres in some neighborhood $U_x \subset M$ of x. Choose a cover M by finitely many neighborhoods, and consider $(\varphi_{x_1}, \ldots, \varphi_{x_k}) : E \to \underline{\mathbb{R}}^n \oplus \ldots \oplus \underline{\mathbb{R}}^n$. This is injective on every fibre.

Step 2: Still for compact M, we show that there is an injective vector bundle map $E^r \to \underline{\mathbb{R}}^{r+\dim(M)}$. For that, we start with an injective vector bundle map $\varphi : E \to \underline{\mathbb{R}}^N$, $N > r + \dim(M) = \dim(E)$. Consider φ just as a map $E \to \mathbb{R}^N$ ($E \to \mathbb{R}^N \times M \to \mathbb{R}^N$), and take $s \in \mathbb{R}^N \setminus \{0\}$ to be a regular value. Then the preimage of s is empty, hence $E \xrightarrow{\varphi} \underline{\mathbb{R}}^N \to \underline{\mathbb{R}}^N/\underline{\mathbb{R}} \cdot s$ is still injective.

Step 3: Given an arbitrary M and a compact $K \subset M$, we can find a vector bundle map $E \to \underline{\mathbb{R}}^{r+\dim(M)}$ which is injective at each point of K. (Just extend in whatever way you like to the whole M).

Step 4: Same as step 3, but where $K = K_1 \cup K_2 \cup \ldots$ is a union of compact subsets K, which have pairwise disjoint open neighborhoods. Then use partition of unity.

Step 5: Every M can be written as a finite union of sets K as in step 4. For this, take a proper function $f : M \to [0, \infty)$ and consider $f^{-1}([0,1] \cup [2,3] \cup [4,5] \cup \ldots)$ and $f^{-1}([1,2] \cup [3,4] \cup [5,6] \cup \ldots)$. This gives an injective map $E \to \underline{\mathbb{R}}^N, N \gg 0$. Then repeat step 2 to reduce it to $N = \text{rank}(E) + \dim(M)$. □

Corollary 26. *There is some F such that $E \oplus F \cong \underline{\mathbb{R}}^{r+\dim(M)}$.*

Corollary 27. *Vector bundles of bounded rank correspond to projective modules of finite rank over $C^\infty(M, \mathbb{R})$ as a category.*

20 April 25

Applications of vector fields to vector bundles Let $\pi : E \to M$ vector bundle. Fix a vector field X on M.

Definition 38. *A linear lift of X is a vector field Y on E which, in a local trivialization, $E|_U \cong \mathbb{R}^r \times U$ has the form $Y_{v,x} = (A(x)v, X_x)$ where $A : U \to \text{Mat}(r \times r, \mathbb{R})$ is smooth.*

Example 42. *If $X = 0, Y$ is just a collection of linear vector fields on the fibers of E. It integrates to a fibrewise linear flow.*

Remark 36. *In local coordinates $(x_1, \ldots, x_n)$, $X = \sum_k \zeta_k(x_1, \ldots, x_n)\partial x_k$, then in local coordinates $(v_1, \ldots, v_r, x_1, \ldots, x_n)$ on E, $Y = \sum_{i,j} A_{ij}(x_1, \ldots, x_n) v_j \partial_{v_i} + \sum_k \zeta_k(x_1, \ldots, x_n)\partial_{x_k}$.*

Remark 37. *One should check that the notion of linear lift is independent of local trivializations. A change of trivialization $(v, x) \xrightarrow{F} (B(x)v, x)$ for some $B(x) \in \text{GL}(r, \mathbb{R})$ yields a transformation rule $DF_{F^{-1}(v,x)}(Y_{F^{-1}(v,x)}) = DF_{F^{-1}(v,x)}(A(x)B(x)^{-1}v, X_x) = (BAB^{-1}v + (DB \cdot X)B^{-1}v, X)$. Note that linearity is indeed invariant (but $A = 0$ is not.)*

If Y_1 and Y_2 are linear lifts of X, then so is $rY_1 + (1-r)Y_2$ for any $r : M \to \mathbb{R}$.

Lemma 28. *Any vector field on M admits a linear lift to E.*

Proof. Partition of unity. □

Remark 38. *The space of all linear lifts of X is an affine space over the space of fibrewise linear vector fields (lifts of $\mathcal{O}$).*

Lemma 29. *Let Y be a linear lift of X. If the flow of X is well-defined on $U \subset \mathbb{R} \times M$, the flow of Y is well-defined over the preimage $V = (\text{id}_\mathbb{R} \times \pi)^{-1}(U) \subset \mathbb{R} \times E$. These flows fit into a diagram*

$$\begin{array}{ccc} \mathbb{R} \times E \supset V & \xrightarrow{\psi} & E \\ \downarrow{\scriptstyle \text{id}\times\pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R} \times M \supset U & \xrightarrow{\varphi} & M \end{array}$$

Just as a reminder, the $\mathbb{R}$ is the time. Moreover, ψ is a linear map between fibres, by which we mean the following: Let $i_0 : M \to \mathbb{R} \times M$, $p : \mathbb{R} \times M \to M$ be inclusion and projection.

Theorem 20.1. *For any vector bundle $E \to \mathbb{R} \times M$, there exists an isomorphism $p^* i_0^* E \to E$ which is the identity on the fibres over $\{0\} \times M$.*

This means that we have a family of linear isomorphisms $\Psi_{t,x} : E_{0,x} \to E_{t,x}$ which are the identity for $t = 0$, and which are smooth in local trivializations.

Proof. Take a linear lift of $X = \partial t$, and take its flow. □

Corollary 28. *Given a vector bundle $E \to N$ and two homotopic maps $f_0, f_1 : M \to N$, we have $f_0^* E = f_1^* E$.*

Proof. Let the homotopy be $\mathbb{R} \times M \xrightarrow{F} N$ such that f_0 and f_1 are the two endpoints. Then $f_0^* E \cong F^* E|_{\{0\} \times M}$ and $f_1^* E \cong F^* E|_{\{1\} \times M}$. Then apply the theorem to $F^* E$. □

Corollary 29. *If M is contractible, any vector bundle over M is trivial.*

Application 22. *Take a vector bundle $E^r \to S^n$. Then E is trivial on each hemisphere. Choosing trivializations and comparing them yields a map $S^{n-1} \to \mathrm{GL}(r, \mathbb{R})$. The homotopy class of this map (up to multiplication by a constant) determines the isomorphism class of the vector bundle.*

Remark 39. *Take $n > 1$ Then one can arrange that the clutching function $S^{n-1} \to \mathrm{GL}(r, \mathbb{R})^+$ and its homotopy class is now unique. In fact, one can achieve that the map takes values in $\mathrm{SO}(r)$.*

Thus vector bundles on spheres correspond to homotopy classes of SO.

Problem 19. *Show that any vector bundle over S^3 is trivial. (Try not to use too much homotopy theory. Use only fundamental groups if you can.) Higher rank: can split off trivial ones. Rank 1: double cover (nothing happens). Rank 2 and 3 are interesting.*

Recall that a Grassmannian $\mathrm{Gr}(r, N)$ is the set of r-dimensional linear subspaces of $\mathbb{R}^N$, which can be described also as the set of symmetric idempotent matrices of rank r and size N. (This shows it's a submanifold of the space of all matrices.)

The trivial vector bundle $\underline{\mathbb{R}}^N \to \mathrm{Gr}(r, N)$ carries a canonical idempotent endomorphism (namely, at A, apply A). Its image is the tautological bundle $\tau \to \mathrm{Gr}(r, N)$.

Lemma 30. *If $E^r \to M^n$, then E is isomorphic to the pullback of τ by some map $M \to \mathrm{Gr}(r, N)$ for any $N \geq r + n$.*

Proof. We know that E is a direct summand of $\underline{\mathbb{R}}^N \to M$. Writing it in that way gives a map $M \to \mathrm{Gr}(r, N)$ (mapping each point to its fibre in E), for which the statement is trivial. □

Lemma 31. *Consider two maps $f_0, f_1 : M \to \mathrm{Gr}(r, N)$ for $N > r + n$. If the vector bundles $f_0^* \tau, f_1^* \tau$ are isomorphic, the maps f_0, f_1 must be homotopic. (Notice the condition on dimension is different from that of above.)*

Proof. This is equivalent to the following statement: given two surjective vector bundle maps $\varphi_0, \varphi_1 : \underline{\mathbb{R}}^N \to E$ over M, there is a surjective vector bundle map $\varphi : \underline{\mathbb{R}}^N \to p^* E$ over $\mathbb{R} \times M$ such that φ restricts to φ_0, φ_1 on the two ends. But that follows from the Sard-style arguments from last time. □

Corollary 30. *Over M^n, vector bundles of rank r up to isomorphism corresponds to maps $M \to \mathrm{Gr}(r, N)$ up to homotopy, where $N > n + r$.*

Remark 40. *We now have two descriptions of vector bundles over S^n: as $\pi_n(\mathrm{Gr}(r, N))$ for $N \gg 0$ and $\pi_{n-1}(\mathrm{SO}(r))$, so these two must be the same. In fact, $\Omega \mathrm{Gr}(r, \infty) \cong O(r)$ where ∞ means taking the limit.*

21 April 27

Problem 20. *M compact manifold, $\varphi : M \to M$ involution, which has an even (finite) number of fixed point. Show that M is a boundary (i.e. trivial in the bordism ring).*

Consider the Grassmannian $\mathrm{Gr}(k, N)$ and its tautological bundle τ. There is a one-point compactification $\tau^+ = \tau \cup \{\infty\}$. Suppose that we are given a continuous map $S^{n+k} \xrightarrow{\varphi} \tau^+$. τ^+ is not a manifold, but we can perturb our map to be smooth outside a small neighborhood of $\varphi^{-1}(\infty)$. Moreover, a further perturbation ensures that φ is transverse to the zero section. Then $M = \varphi^{-1}(\mathrm{Gr}(k, N) \subset \tau^+) \subset S^{n+k}$ is a smooth n-manifold. Moreover, its normal bundle in S^{n+k} is isomorphic to $(\varphi|_M)^*\tau$.

Now let's turn this construction around.

Pontryagin-Thom construction Given any compact n-manifold M, take an embedding $M \hookrightarrow S^{n+k}$ for sufficiently large k, the normal bundle is isomorphic to the pullback of τ by some map $M \to \mathrm{Gr}(k, N)$ for some large N. Using the tubular neighborhood theorem, we extend this to a proper map $U \to \tau$ where $U \subset S^{n+k}$ is a neighborhood of M. Then extend this to $S^{n+k} \to \tau^+$ by sending all of $S^{n+k} \setminus U$ to ∞.

Notice these embeddings are not unique; they depend on the embedding of M chosen, but we have the following:

Theorem 21.1 (Pontryagin-Thom). *$\Omega_n \cong \pi_{n+k}(\tau^+ \subset \mathrm{Gr}(k, N))$ provided that $k > n, N > k + n$.*

The RHS is independent of N (directly checkable), and is independent of k due to stable homotopy theory.

Corollary 31. *Ω_n is finite in each dimension.*

Definition 39. *Let M be a compact n-manifold. We say M is stably parallizable if $TM \oplus \underline{\mathbb{R}}^r \cong \underline{\mathbb{R}}^{r+n}$ for some r. In particular, spheres (more generally, hypersurfaces) are parallizable.*

Suppose that M is stably parallizable. Then, for any embedding $M \hookrightarrow \mathbb{R}^{n+k}$, the normal bundle is also stably trivial, since $\nu \oplus TM \cong \underline{\mathbb{R}}^{n+k} \implies \nu \oplus \underline{\mathbb{R}}^{r+n} \cong \nu \oplus TM \oplus \underline{\mathbb{R}}^r \cong \underline{\mathbb{R}}^{n+k+r}$. Hence, we can (after increasing k) embed $M \hookrightarrow S^{n+k}$ such that the normal bundle is actually trivial. Apply the Pontryagin-Thom construction to this embedding, then the map to $\mathrm{Gr}(k, N)$ is a constant, and the map $U \to \tau$ lies in a single fibre of τ. Hence the whole map $S^{n+k} \to S^k \hookrightarrow \tau^+$.

For instance, any stably parallelizable 4-manifold is a boundary of a 5-manifold, since the fourth stably homotopy group of sphere is trivial.

In general, given a map $S^{n+k} \to (\tau \times X)^+$ for some compact X, let U be the part that doesn't map to infinity. Take the projection to τ and then preimage of the zero section, then we get a manifold M. On the other hand, by projecting to X we get a map $M \to X$. The converse is also true, and thus we have $\Omega_n(X) \cong \pi_{n+k}((\tau \times X)^+)$.

Invariants of vector bundles Let M be a compact manifold. $E \to M$ a rank r vector bundle. Let $s : M \to E$ be a section that is transverse to the zero section. Then $Z = s^{-1}(M \subset E)$ is a submanifold of dimension $n - r$.

Definition 40. *The unoriented Euler class $e(E)$ is $[Z] \in \Omega_{n-r}(M)$. This is an invariant of E (we can interpolate sections, etc).*

We have the following properties:

- $e(E \oplus \underline{\mathbb{R}}) = 0$. (Consider the unit $\underline{\mathbb{R}}$-section.)
- $e(E \oplus F) = e(E)e(F)$ where we use the intersection product.

Example 43. *$M = \mathbb{RP}^n = \mathrm{Gr}(1, n + 1)$, $E = \tau$ the tautological bundle, then $e(E) = [\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n]$ the hyperplane inclusion. Therefore, $e(nE) = [* \to \mathbb{RP}^n] \neq 0$, thus nE is nontrivial.*

(Notation remark: $nE = \underbrace{E \oplus E \oplus \ldots \oplus E}_{n \text{ copies}}$, and $ne(E) = \underbrace{e(E) \cdot e(E) \cdot \ldots \cdot e(E)}_{ncopies}$, where $\cdot$ is the intersection product.)

Remark 41. *If M and E are oriented, we can define the oriented Eulcer class $e(E) \in \Omega^{\mathrm{SO}}_{n-r}(M)$.*

The simplest special case is that if L is a real line bundle, $e(L) \in \Omega_{n-1}(M)$.

Definition 41. *The (bordism) first Stiefel-Whitney class of $E \to M$ is $w_1(E) = e(\det(E)) \in \Omega_{n-1}(M)$.*

We have the following properties:

- $w_1(E \oplus \underline{\mathbb{R}}) = w_1(E)$.
- $w_1(E \oplus F) = w_1(E) + w_2(F)$. This is harder to prove.

How do we get higher SW classes?

Grothendieck Construction Take $E^r \to M^n$ vector bundle. There is an associated bundle of real projective spaces $\mathbb{RP}(E) \to M$, the fibre of this at any point $x \in M$ is the projective space of E_x. Moreover, this carries a topological line bundle $\tau \to \mathbb{RP}(E)$.

For any k, consider $e((r+k-1)\tau)$. Its image under $\mathbb{RP}(E) \to M$ is a class in $\Omega_{n-k}(M)$. All these classes are invariants of the bundle E. For line bundle E, these are just powers of $e(E) = w_1(E)$.

22 May 02

M^n compact manifold, $E^r \to M$ vector bundle, $\mathbb{RP}(E) \to M^n$ bundle of projective spaces. (A point in $\mathbb{RP}(E)$ is a pair (x, λ) consisting of $x \in M$ and a 1-d linear subspace $\lambda \subset E_x$).

$\tau_E \to \mathbb{RP}(E)$ tautological line bundle. (The fibre of τ_E over (x, λ) is exactly λ, it is a subbundle of the pullback of E to $\mathbb{RP}(E)$).

Hence we have $e(\tau_E) = [\text{zero set of a generic section}] \in \Omega_{n+r-2}(\mathbb{RP}(E))$. More generally, $ie(\tau_E) = e(i\tau_E) = [\text{common zero set of } i \text{ generic sections of } \tau_E] \in \Omega_{n+r-1-i}(\mathbb{RP}(E))$.

Definition 42. *$u_i(E) \in \Omega_{n-i}(M)$ is the image under $\mathbb{RP}(E) \to M$ of $(i + r - 1)e(\tau_E)$.*

These bordism classes are invariants of the vector bundle E. (We can call them the "bordism-theoretic complementary Stiefel-Whitney classes".)

Example 44. *If E is a line bundle, $u_i(E) = ie(E)$.*

Theorem 22.1. *$u_i(E) = u_i(E \oplus \underline{\mathbb{R}})$.*

Proof. For simplicity, let's equip E with an inner product, and $E \oplus \underline{\mathbb{R}}$ with the corresponding inner product. Then, $\tau_{E\oplus\underline{\mathbb{R}}}$ has a distinguished section, which is given by projecting the unit section of $\underline{\mathbb{R}}$ to the tautological line. This section vanishes exactly along $\mathbb{RP}(E) \subset \mathbb{RP}(E \oplus \underline{\mathbb{R}})$. Therefore, $e(\tau_{E\oplus\underline{\mathbb{R}}}) \cdot \ldots \cdot e(\tau_{E\oplus\underline{\mathbb{R}}})$ (j of them) equals $[\mathbb{RP}(E) \hookrightarrow \mathbb{RP}(E \oplus \underline{\mathbb{R}})] \cdot$ ($j - 1$ of them) = image of ($j - 1$ of them) under the injection map. Now, projection to M establishes the desired result. ☐

One example: suppose $M = \partial W$, then we have $0 \to TM \to TW|_M \to \underline{\mathbb{R}} \to 0$, so $TM \oplus \underline{\mathbb{R}} \cong TW|_M$. In order to study $u_i(TM)$, it is therefore sufficient to study $u_i(TW|_M)$.

Given a partition $n = i_1 + \ldots + i_k$, define $u_{i_1,\ldots,i_k}(M) = u_{i_1}(TM) \cdot \ldots u_{i_k}(TM)$ mapped to $\Omega_0 \cong \mathbb{Z}/2$. These are called complementary Stiefel-Whiteney numbers. (They are invariants of manifold, and are in fact homotopy invariants.)

Theorem 22.2. *If M is the boundary of a compact manifold, then $u_{i_1,\ldots,i_k}(M) = 0$.*

Hence, in fact, $u_{i_1,\ldots,u_k} : \Omega_n \to \mathbb{Z}/2$ are group homomorphisms. Also, though we'll not prov it, this is in fact an iff statement.

Proof. Let $M = \partial W$, and consider $u_i(TM) = u_i(TM \oplus \underline{\mathbb{R}}) = u_i(TW|_M)$, which is the intersection of the same map to W with M. Therefore, the whole product is the boundary of a compact 1-manifold mapping to W. $\square$

Here's an example. $M = \mathbb{RP}^n$, τ the tautological bundle. Let TM_ℓ be ways to move ℓ around in $\mathbb{R}^{n+1}$ to first order, i.e. $\mathrm{Hom}(\tau, \mathbb{R}^{n+1}/\tau)$. Hence we have $0 \to \mathrm{Hom}(\tau,\tau) \hookrightarrow \mathrm{Hom}(\tau, \mathbb{R}^{n+1}) \to TM \to 0$. In other words, $TM \oplus \underline{\mathbb{R}} \cong \tau \oplus \ldots \oplus \tau = \tau \otimes \underline{\mathbb{R}}^{n+1}$. Take $E = TM \oplus \underline{\mathbb{R}} = \tau \otimes \underline{\mathbb{R}}^{n+1}$, then $\mathbb{RP}(E) \cong \mathbb{RP}^n \times \mathbb{RP}^n$. (This follows from $\mathbb{RP}(V) \cong \mathbb{RP}(V \otimes W)$ for any 1-d W.) Then τ_E over $\mathbb{RP}(E)$ is τ over the first $\mathbb{RP}^n$ tensor the τ over the second one. Therefore, $e(\tau_E) = [\mathbb{RP}^{n-1} \times \mathbb{RP}^n] + [\mathbb{RP}^n \times \mathbb{RP}^{n-1}] \in \Omega_{2n-1}(\mathbb{RP}^n \times \mathbb{RP}^n)$. Then j fold product of $e(\tau_E) = \sum_{p+q=j} \binom{j}{p} [\mathbb{RP}^{n-p} \times \mathbb{RP}^{n-q}] \in \Omega_{2n-j}(\mathbb{RP}^n \times \mathbb{RP}^n)$. Project and get $u_i(E) =$ projection of $i+n$ copies of $e(\tau_E) = \binom{i+n}{i} [\mathbb{RP}^{n-i}] + \text{junk} \in \Omega_{n-i}(\mathbb{RP}^n)$ where for the junk part we mean bordism classes which have lower dimensional image.

For $n = 2$, i.e. $M = \mathbb{RP}^2$, $u_1(TM) = \binom{3}{1}[\mathbb{RP}^1] = [\mathbb{RP}^1]$, $u_2(TM) = \binom{4}{2}[\mathbb{RP}^0] = 0$. Hence $u_2 = 0, u_{11} = u_1(TM) \cdot u_1(TM) = [\mathbb{RP}^1] \cdot [\mathbb{RP}^1] = 1$. This gives another proof that $\mathbb{RP}^2$ is not a boundary.

Problem 21. *Compute $u_{i_1,\ldots,i_k}$ for $\mathbb{RP}^3$ and $\mathbb{RP}^4$. (3: u_3, u_{12}, u_{111}, 4: $u_4, u_{13}, u_{22}, u_{112}, u_{1111}$).*

23 May 04

Let M^n be a manifold. A *stable framing* of M is an isomorphism of vector bundles $TM \oplus \underline{\mathbb{R}}^{N-n} \cong \underline{\mathbb{R}}^N$. Two stable framings are considered the same if they differ by extending further $\underline{\mathbb{R}}$s. Stable framings that can be deformed into each other are also equivalent.

Lemma 32. *Given a stable framing over M, the set of all possible stable framings is $[M, O(N)]$ for $N \gg 0$, i.e. the homotop classes of maps $M \to O(N)$.*

Example 45. *$*$ has two stable framings, as $\pi_0(O(N)) = \mathbb{Z}/2$. Similarly, S^1 has four stable framings, with two compatible with each orientation, since $\pi_1(O(N)) = \mathbb{Z}/2$. ($\pi_1(SO(2)) = \mathbb{Z}, \pi_1)SO(N)) = \mathbb{Z}/2$ for $N > 2$.)*

The two framings of S^1 are the trivial framing (the usual trivialization of TS^1) and the nontrivial framing (where everywhere $TM \oplus \underline{\mathbb{R}}$ has the same "direction"; need a picture here.) For the trivial framing, add the unit normal bundle to the trivialization TS^1, and see the resulting bundle is not isomorphic to the nontrivial framing.

Recall that if $M = \partial W$ for some manifold W, then we have $TM \oplus \underline{\mathbb{R}} \cong TW|_M$ (where the $\underline{\mathbb{R}}$ is the "choice of pointing outward vector"; this choice is not canonical nor unique, but any two choices can be deformed into each other.) Therefore a stable framing of W induces a stable framing of M. We can therefore define the stably framed bordism group Ω_n^{fr}.

Example 46. $\Omega_0^{\mathrm{fr}} = \Omega_0^{\mathrm{SO}} = \mathbb{Z}$. *For each line segment, the $\underline{\mathbb{R}}$ factor points towards one end, so the framing on the point on that end becomes "+1" and the other end becomes "-1".*

Example 47. $\Omega_1^{\mathrm{fr}} = \mathbb{Z}/2$.

To show this we can appeal to the classification of 1-d manifolds. There are two framings up to orientation for S^1, and the nontrivial framing obviously bounds (one can obviously extend that nontrivial framing to the entire disk), and two trivial framings bound a cylinder, so it remains to show that the trivial framing doesn't bound.

For that, take W to be a genus g oriented surface with one boundary compact. By hand, one can find a framing of W which restricts to the nontrivial framing on $\partial W = S$. On the other hand, framings of W are parametrized by $[W, SO(N)] \cong \mathrm{Hom}(\pi_1(W), \pi_1(SO(N)))$ (by cellular approximation), which is isomorphic to $\mathrm{Hom}(\pi_1(W), \mathbb{Z}/2)$. On the other hand, the framing of ∂W is parametrized by $[\partial W, SO(N)] \cong \mathrm{Hom}(\pi_1(\partial W), \mathbb{Z}/2)$, and we have an obvious diagram:

$$\begin{array}{ccc} [W, SO(N)] & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(W), \mathbb{Z}/2) \\ \downarrow{i_*} & & \downarrow{i_*} \\ [\partial W, SO(N)] & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(\partial W), \mathbb{Z}/2) \end{array}$$

where both sides are induced by the inclusion $i_* : \partial W \to W$. However, the RHS map is zero, because $\mathbb{Z}/2$ is abelian so we have $\mathrm{Hom}(\pi_1(X), \mathbb{Z}/2) = \mathrm{Hom}(H_1(X), \mathbb{Z}/2) = H^1(X; \mathbb{Z}/2)$, so the map is $H^1(W; \mathbb{Z}/2) \to H^1(\partial W; \mathbb{Z}/2)$, whose image is the kernel of $H_0(\partial W; \mathbb{Z}/2) \cong H^1(\partial W; \mathbb{Z}/2) \to H^2(W, \partial W; \mathbb{Z}/2) \cong H_0(W; \mathbb{Z}/2)$, which is trivial.

Therefore, any two framings of W (compatible with orientation) induces the same framing of ∂W, thus ∂W is always nontrivially framed.

Example 48 (Outline of the 2-d case)**.** $\Omega_2^{\mathrm{fr}} \cong \mathbb{Z}/2$.

Historical remark: Pontryagin made a mistake here by thinking this object is zero. To see how to derive the statement above, let M be an oriented surface. A stable framing of M leads to a quadratic refinement of the mod 2 intersection pairing, by which we mean a quadratic form $q : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ such that $q(x + y) - q(x) - q(y) = x \cdot y$, where the RHS is the intersection product.

Of course, in general, given any refinement of $\cdot$, any affine shifting of the refinement is again a refinement. But given a class in $H_1(M, \mathbb{Z}/2)$, one can represent it by an embedded circle in M, and then restrict the stable framing of M to the loop, giving $q(x) \in \Omega_1^{\mathrm{fr}} \cong \mathbb{Z}/2$. (Reference: [Johnson 1980, *Spin Structures and Quadratic forms on Surfaces*.]) Then the map $\Omega_2^{\mathrm{fr}} \to \mathbb{Z}/2$ is given by the Arf invariant $\mathrm{Arf}(q)$.

In fact, Ω_*^{fr} was implicitly discovered before the normal bordism classes! Here's why. Consider a map $S^n \xrightarrow{f} S^{N-n}$. The preimage of a regular value is a manifold $M^n \subset S^N$ whose normal bundle comes with a preferred trivialization. Hence, M comes with a preferred stable framing

$$TN \oplus \underbrace{\nu M \oplus \underline{\mathbb{R}}}_{\text{trivial}} \cong \underbrace{TS^N|_M \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{N+1}}_{\text{framing of } S^N}$$

This yields a map from stable homotopy groups to stable framing bordism groups

$$[S^N, S^{N-n} \cong \pi_N(S^{N-n})] \to \Omega_n^{\mathrm{fr}}$$

Theorem 23.1 (Pontryagin-Thom)**.** *This is an isomorphism for $N \geq 2n - 1$.*

Here's how to construct the reverse direction map: given M with a stable framing, embed it into $\mathbb{R}^N \subset S^N$ for large N with trivial normal bundle. Map tubular neighborhood to $\mathbb{R}^{N-n}$ (with $M \mapsto 0$) and the rest of the space going to ∞.

Example 49. $\pi_4(S^3) = \mathbb{Z}/2$.

One can also introduce $\Omega_*^{\mathrm{fr}}(X)$, which is the bordism of stably framed manifolds M with a map $M \to X$.

Theorem 23.2. $\Omega_n^{\mathrm{fr}}(X) \cong \pi_N((\mathbb{R}^{N-n} \times X)^+)$ *for $N \gg 0$. The RHS is called the stable homotopy groups of X (or X^+).*

Remark 42. *We have canonical forgetful maps $\Omega_*^{\mathrm{fr}} \to \Omega_*^{\mathrm{SO}} \to \Omega_*$. Unfortunately, these maps always vanish in positive degrees and are therefore uninteresting. There is an interesting alternative, the spin bordism Ω_*^{spin}, such that $\Omega_*^{\mathrm{fr}} \to \Omega_*^{\mathrm{spin}}$ is (almost) computable and nontrivial. Unfortunately we don't have time to cover that in this course.*

Preview for the next class: take $p : M^m \to N^n$ a proper submersion. We'll define $\Omega_*^{\mathrm{fr}}(N) \to \Omega_{*+m-n}^{\mathrm{fr}}(M)$ (note that this is a shriek map) called the *Becker-Gottieb transfer map.* This is a map "in the wrong direction" and is rather difficult to construct in pure AT setting; but the differential story is more pleasant. In particular, if N is connected, $\Omega_0^{\mathrm{fr}}(N) = \mathbb{Z}$ and we get a distinguished class in $\Omega_{m-n}^{\mathrm{fr}}(M)$, which are generalizations of Euler classes.

24 May 09

Euler characteristics Let M be a compact manifold, then $\chi(M) \in \mathbb{Z}$.

Definition 43. *Let's recall its definition.*

- *One could embed* $M \hookrightarrow \mathbb{R}^N$, *and let* $r : U \to M \subset U$ *a retraction of a local neighborhood, then* $\chi(M) = L(r)$ *is the Lefschetz fixed point number of* r.
- *Or, we can define* $\chi(M) = L(\mathrm{id}_M)$. *(For this, we perturb* id_M *to a self-map* f *with nondegenerate fixed points, and* $L(f) = \sum_{f(x)=x} \operatorname{sgn}\det(1 - D_x f)$, *where* $1 - D_x f$ *is an endomorphism of* TM_x*).*
- $\chi(M) \bmod 2$ *is the image of the Euler class of the tangent bundle* $e(TM) \in \Omega_0(M)$ *under the map* $\Omega_0(M) \to \Omega_0(*) = \mathbb{Z}/2$. *This counts the zeros mod 2 of a vector field transverse to the zero section. If* ξ *is such a vector field, and* $x \in M$ *is a point at which* ξ *vanishes, then* $D\xi_x : TM_x \to TM_x$ *is an isomorphism. Then* $\chi(M) = \sum_{\xi(x)=0} \operatorname{sgn}\det(D\xi_x)$.

Let's see 2 and 3 are the same: consider a flow of short time, then the fixed points of the flow are exactly where ξ_x vanishes. Additionally, $D_x f$ is the exponential of $D\xi_x$, so the equivalence follows.

Remark 43. *If* M *is odd-dimensional, then* $\chi(M) = 0$. *(Consider* $L(f)$ *where* f *is close to identity; if we multiply by* $L(f^{-1})$, *we multiply by* $(-1)^n$, *but we're supposed to have the same value.)*

Now, let $p : M \to N$ be a submersion between compact manifolds. Then each fibre $p^{-1}(y) = M_y$ is itself a manifold, and thus family of manifolds is locally trivial (Ehresmann's theorem). Consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{id} & N \end{array}$$

Let x be a fixed point of f, $y = p(x)$. Then

$$\begin{array}{ccc} TM_x & \xrightarrow{Df_x} & TM_x \\ \downarrow{\scriptstyle Dp_x} & & \downarrow{\scriptstyle Dp_x} \\ TN_y & \xrightarrow{id} & TN_y \end{array}$$

Then $\mathrm{id} - Df_x : TM_x \to \ker(Dp_x) \subset TM_x$. Locally speaking, if $f(z, y) = (h(z, y), y)$ (fibre, base) then $(id - f)(z, y) = (h(z, y) - z, 0)$.

We say that x is nondegenerate if $\operatorname{im}(id - Df_x) = \ker(Dp_x)$. For a generic f, this holds for all fixed points. Then $\operatorname{Fix}(f) \subset M$ is a submanifold of dimension $\dim(N)$. (But $p|_{\operatorname{Fix}(f)} : \operatorname{Fix}(f) \to N$ is not a submersion.) (Locally speaking: x is a nondegenerate value iff 0 is a regular value of $h(z, y) - z$.)

Note that the tangent space $T(\operatorname{Fix}(f)_x) = \ker(id - Df_x) \implies T(\operatorname{Fix}(f)_x) \oplus \ker(Dp_x) \cong TM_x \cong TN_y \oplus \ker(Dp_x)$. Hence, as stable vector bundles we have $T(\operatorname{Fix}(f)) \cong p^*TN|_{\operatorname{Fix}(f)}$.

Suppose that N is stably framed (TN is stably trivial) Then so is $\operatorname{Fix}(f)$ and we get an invariant $L(f) = [\operatorname{Fix}(f)] \in \Omega^{\mathrm{fr}}_{\dim(N)}$. This is called Dold's parametrized fixed point number.

Example 50. $N = S^1$. $L(f) \in \Omega^{\mathrm{fr}}_1 = \mathbb{Z}/2$.

In particular, for $f = id$, $\chi(M \to N) = L(id) \in \Omega^{\mathrm{fr}}_{\dim(N)}$. This is the parametrized Euler characteristic of Becker and Gottlieb.

There is also an equivalent definition of $\chi(M \to N)$ using vector fields in $\ker(Dp) \subset TM$.

In general (if N is not stably framed, or not even compact), we get $L(f) : \Omega^{\mathrm{fr}}_*(N) \to \Omega^{\mathrm{fr}}_*(M)$ (Dold fixed point transfer) and $\chi(M \to N) : \Omega^{\mathrm{fr}}_*(N) \to \Omega^{\mathrm{fr}}_*(M)$ (Becker-Gottlieb transfer).

Definition of the fixed point transfer: take a class in $\Omega_k^{\mathrm{fr}}(N)$, represented by some $q : Q \to N$, where Q is stable framed. Consider the pullback $q^*M \to Q$, where the first is $\{(z,x) \subset Q \times M : q(z) = p(x)\}$. q^*f acts on q^*M, then consider $[\mathrm{Fix}(q^*f)] \in \Omega_k^{\mathrm{fr}}(q^*M) \to \Omega_k^{\mathrm{fr}}(M)$.

Now let's talk about something else. Reference: [Zhang, *A Counting Formula for the Kervaire semi-characteristic*, Topology (2000)] or [Zizhou, *Bordism theory and the semi-characteristic*, Sci. China (2002)].

Suppose M is a compact manifold, oriented connected, $\chi(M) = 0$.

Lemma 33. *If $\chi(M) = 0$, then M admits a nowhere vanishing vector field.*

Sketch of Proof. Take a vector field ξ transverse to 0. wlog all zeroes of ξ are contained in (the interior of) a single coordinate ball $B^n \subset M$. Now look at $\xi : B^n \to \mathbb{R}^n$, $\xi(x) \neq 0$ on $\partial B^n = S^{n-1}$. Consider $\xi/\|\xi\| : S^{n-1} \to S^{n-1}$. If this is homotopy to a constant, then we can find another map $B^n \to \mathbb{R}^n$ with the same boundary values and which has no zeroes. But $[S^{n-1}, S^{n-1}] = \mathbb{Z}$, and this is actually (when one counts it out) the counting of zeros of ξ. □

Lemma 34. *If $\chi(M) = 0$, M admits a nowhere zero section of T^*M.*

Fix ξ section of T^*M which is nowhere zero. Take $E = \ker(\xi) \subset TM$ which is an oriented subbundle of rank $(n-1)$. (Now can we get a section of E that is nowhere zero?) Let η be a section of E which is transverse to the zero section. The zero-set of η is a finite union of circles. Let C be one of these circles. Then $D\eta : \nu_C \to E|_C$ is an isomorphism of vector bundles over C. Note that $\xi : TM \to \mathbb{R}$ satisfies $\xi \circ \eta = 0$ by definition.

Claim $\xi^* : TM \to \mathbb{R}$ is nonzero on TC. (This is iff TC is nowhere contained in $E|_C$.) (Proof?)

Then $E|_C \to TM|_C \to \nu_C$ is an isomorphism. Combining this and the D_ν isomorphism before gives an automorphism of $E|_C$, hence an element of $\pi_1(\mathrm{GL}_{n-1}(\mathbb{R})) \to \mathbb{Z}/2$. This defines $o(C) \in \mathbb{Z}/2$.

Definition 44. $\kappa(M) = \#\{C : o(C) = 0\} \in \mathbb{Z}/2$.

Theorem 24.1. *This is an invariant of M. (This is "the next invariant" after Euler characteristic.)*

Let's do an example. Let $M = N \times S^1$ where S^1 has coordinate t, $\xi = dt$, and E is TN pulled back to M. η a generic section of TN pulled back to M. If $p(x) = 0$, then η vanishes on $C = \{x\} \times S^1$, $o(C) = 0$, so $\kappa(M) = \chi(N) \bmod 2$. It's only interesting when N has dimension being a multiple of 4, but that case is interesting. (Notice that multiplying by S^1 kills the Euler characteristic.)

Theorem 24.2. *$\kappa(M) = 0$ when $n \equiv 2, 3 \pmod 4$, $\kappa(M) = \sum_i \dim H^{2i}(M;\mathbb{R})$ when $n \equiv 1 \pmod 4$ (the Kervaire semi-characteristic), and $\mathrm{sign}(M)/2$ when $n \equiv 0 \pmod 4$ (the signature).*